Twisted quasi-elliptic cohomology and a twisted character map

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(jt. with Zhen Huan)

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Definition

The first step in defining quasi-elliptic cohomology is to define the group

$$\Lambda_G(g) := C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle.$$

Let $\Lambda_G(g)$ act on X^g via

$$[h, t] \cdot x := h \cdot x.$$

Definition (Huan)

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Equivalently, this is the orbifold K-theory of the groupoid

$\coprod_{[g] \in G_{conj}} X^g /\!\!/ \Lambda_G(g)$, which is the subgroupoid of constant loops in

$$Loop(X /\!\!/ G) := \coprod_{[g] \in G_{conj}} Map_G(P_g, X) /\!\!/ Aut(P_g),$$

where P_g is the principal G-bundle over S^1 with monodromy g.

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$$\begin{aligned} H^3(BG; U(1)) &\xrightarrow{ev^*} H^3(S^1 \times Map(S^1, BG); U(1)) \\ &\xrightarrow{\pi_*} H^2(Map(S^1, BG); U(1)) \end{aligned}$$

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So, the image of α under τ may be written as

$$\tau(\alpha) = (\tau(\alpha)_g)_{[g] \in G_{conj}} \in \prod_{[g] \in G_{conj}} H^2(BC_G(g); U(1)).$$

Fact: A 2-cocycle $\theta : G \times G \to U(1)$ determines a $U(1)$-central extension of G

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$${}^{\theta}K_G(X) \subset K_{G^{\theta}}(X)$$

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Definition (Huan-S.)

Let $\alpha \in H^3(BG; U(1))$. The α-twisted quasi-elliptic cohomology of X is defined as

$${}^\alpha QEll_G(X) := \prod_{[g] \in G_{conj}} {}^{\tau(\alpha)_g} K_{\Lambda_G(g)}(X^g).$$

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Lemma (Huan-S.)

For each $g \in G$, there is a homomorphism

$$p_g : S^1 \times C_G^{\tau(\alpha)_g}(g) \longrightarrow \Lambda_G^{\tau(\alpha)_g}(g)$$

sending $(t, (a, h))$ to $[Nt, (a, h)]$, where N is the order of $(0, g)$ in $C_G^{\tau(\alpha)_g}(g)$.

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This induces a homomorphism

$${}^{\tau(\alpha)_g}K_{\Lambda_G(g)}(X^g) \longrightarrow {}^{\tau(\alpha)_g}K_{\mathbb{T}\times C_G(g)}(X^g)$$

by pulling back the action of $\Lambda_G(g)$ along p_g.

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Theorem (Atiyah-Segal)

There is a natural isomorphism

$$K_G(X) \otimes \mathbb{C} \cong \prod_{[g] \in G_{conj}} (K(X^g) \otimes \mathbb{C})^{C_G(g)}$$

given in the factor corresponding to $[g]$ by

$$V \mapsto \sum_{\xi} \xi \, (V|_{X^g})_{\xi}.$$

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&\xrightarrow{p^*\otimes\mathbb{C}} \prod_{[g]} {}^{\tau(\alpha)_g}K_{S^1\times C_G(g)}(X^g)\otimes\mathbb{C} \\
&\cong \prod_{[g]} {}^{\tau(\alpha)_g}K_{C_G(g)}(X^g)\otimes\mathbb{C}[q^{\pm}] \\
&\overset{A-S}{\cong} \prod_{[g]}\prod_{[h]\in C_G(g)_{conj}} \left(K(X^{g,h})\otimes\tau^2(\alpha)_{g,h}\otimes\mathbb{C}[q^{\pm}]\right)^{C_G(g,h)} \quad (Adem, Ruan) \\
&\overset{Chern}{\cong} \prod_{\substack{[g,h]\\ gh=hg}} \left(H(X^{g,h})\otimes\tau^2(\alpha)_{g,h}\otimes\mathcal{O}^{hol}(L)\right)^{C_G(g,h)} =: {}^{\alpha}Ell_G^{Dev}(X)
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In the last line, L is the space of based lattices in $\mathbb{C}$, and a polynomial $p\in\mathbb{C}[q^{\pm}]$ is sent to $(t_1,t_2)\mapsto p(e^{2\pi i t_1/t_2})$.

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Now, since $p^* \otimes \mathbb{C}$ is injective, we have the following result.

Theorem (Huan-S.)

${}^{\alpha}QEll_G(X)$ is isomorphic to the subgroup of ${}^{\alpha}Ell_G^{Dev}(X)$ generated by elements of the form

$$\prod_{[g,h]} \bigoplus_{n} \bigoplus_{\xi(h)} \xi(h)\, ch((E_g)_n|_{X^{g,h},\xi}) \otimes q^{n/|(0,g)|}$$

where $(E_g) \in {}^{\alpha}QEll_G(X)$.