New constructions of equivariant elliptic cohomology

Matthew Spong

Supervisor: Nora Ganter

University of Melbourne

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Grojnowski's equivariant elliptic cohomology

Let $E_\tau := \mathbb{C}/\langle \tau, 1 \rangle$ be the complex elliptic curve corresponding to $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \operatorname{im} z > 0\}$. Denote the holomorphic structure sheaf by $\mathcal{O}_{E_\tau}$.

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Grojnowski's equivariant elliptic cohomology is a functor $\mathcal{E}\ell\ell_{S^1,\tau}$ defined on finite S^1-CW complexes, and taking values in coherent sheaves of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{O}_{E_\tau}$-algebras. The functor satisfies axioms analogous to those for a cohomology theory taking values in graded rings.

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For a finite S^1-CW complex X and a point $a \in E_\tau$, let

$$X^a = \begin{cases} X^{\mathbb{Z}/n\mathbb{Z}} & |a| = n \\ X^a = X^{S^1} & \text{else.} \end{cases}$$

Grojnowski's S^1-equivariant elliptic cohomology

Fact: For any finite T-complex X there exists an open cover $\{U_a\}_{a \in E_\tau}$ of E_τ such that, whenever U_a intersects with U_b, we have either $X^a \subset X^b$ or $X^b \subset X^a$.

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Definition

The value of the sheaf $\mathcal{E}\ell\ell_{S^1,\tau}(X)$ on U_a is the $\mathcal{O}_{E_\tau}(U_a)$-algebra

$$H_{S^1}(X^a) \otimes_{\mathbb{C}[u]} \mathcal{O}_{E_\tau}(U_a - a),$$

where a polynomial in $\mathbb{C}[u]$ is regarded as a holomorphic function on a small neighbourhood $U_a - a$ of $0 \in E_\tau$ via $\mathbb{C} \twoheadrightarrow \mathbb{C}/\langle \tau, 1 \rangle$.

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The gluing maps on $U \subset U_a \cap U_b$ are induced by inclusion $X^b \subset X^a$, followed by an isomorphism induced by translation $t_{b-a} : U - b \to U - a$.

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An element $f \in \mathcal{O}_{E_\tau}(U_a)$ acts by multiplication on the right factor by $t_a^* f$. By definition, we have $\mathcal{E}\ell\ell_{S^1,\tau}(*) = \mathcal{O}_{E_\tau}$.

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Definition

Let (P, A) be a principal T-bundle P on E_τ with flat connection $A \in \Omega^1(P) \otimes \mathfrak{t}$. Two connections $(A, P) \sim (A', P')$ are *gauge equivalent* if there is an isomorphism $\Phi : P \to P'$ of bundles such that $A = \Phi^* A'$.

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given by taking the holonomy of (P, A) around a loop in E_τ. Since A is flat, this only depends on the homotopy class of the loop, and defines a bijection between $E_{T,\tau}$ and the set $\{(P, A)\}$, modulo gauge equivalence.

Principal bundles on $\mathbb{T}^2$

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$$\begin{array}{ccc} P & \xrightarrow{\phi} & P \\ \downarrow & & \downarrow \\ \mathbb{T}^2 & \xrightarrow{(A,s)} & \mathbb{T}^2 \end{array}$$

covering diffeomorphisms $(A, s) : r \mapsto Ar + s$, where $(A, s) \in \mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2$.

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covering diffeomorphisms $(A, s) : r \mapsto Ar + s$, where $(A, s) \in \mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2$. For a T-space X, this acts by precomposition on T-equivariant maps

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \\ \mathbb{T}^2 & & \end{array}$$

Maximal torus of $\mathcal{G}_{ext}$ and Weyl group

This may be written as the semidirect product

$$\mathcal{G}_{ext}(P) = (\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2) \ltimes \mathrm{Map}(\mathbb{T}^2, T),$$

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acting on $\mathrm{Map}_T(P, X) = \mathrm{Map}(\mathbb{T}^2, X)$. There is a maximal torus

$$\mathbb{T}^2 \times T \subset \mathcal{G}_{ext}(P).$$

The associated (elliptic) Weyl group is

$$W = \mathrm{GL}_2(\mathbb{Z}) \ltimes \check{T}^2$$

where $\check{T}^2 \subset \mathrm{Map}(\mathbb{T}^2, T)$.

Maximal torus of $\mathcal{G}_{ext}$ and Weyl group

Consider the Weyl action of $(m, n) \in \check{T}^2$ on the complex Lie algebra $\mathbb{C}^2 \times \mathfrak{t}_{\mathbb{C}}$ of the maximal torus. The action is free, given by

$$(t_1, t_2, x) \mapsto (t_1, t_2, x + mt_1 + nt_2).$$

Restricting to $t_1 = \tau$ and $t_2 = 1$, the quotient by this action is

$$\mathfrak{t}_{\mathbb{C}}/(\check{T}\tau + \check{T}) \cong \check{T} \otimes (\mathbb{C}/\langle \tau, 1 \rangle) =: E_{T,\tau}.$$

Definition

Define

$$E_T := (\mathbb{H} \times \mathfrak{t}_{\mathbb{C}})/\check{T}^2.$$

Thus, the fiber of the natural projection $E_T \to \mathbb{H}$ over τ is $E_{T,\tau}$.

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Thus, the fiber of the natural projection $E_T \to \mathbb{H}$ over τ is $E_{T,\tau}$. Furthermore, the action of $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{Z})$ descends to $\tau \mapsto (a\tau + b)/(c\tau + d)$ on $\mathbb{H}$.

What do loops have to do with elliptic cohomology?

Given a finite T-space X, we want to construct a $\check{T}^2$-equivariant sheaf over $\mathbb{H} \times \mathfrak{t}_{\mathbb{C}}$ which is cohomological in X. The hope is that this will be $\mathcal{E}\ell\ell_T(X)$.

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Borel equivariant cohomology $H_T(X;\mathbb{C})$ is a good candidate, since this has a $H_T = H_T(*;\mathbb{C})$-algebra structure induced by $X \to *$, and H_T is the ring of polynomial functions on $\mathfrak{t}_{\mathbb{C}}$.

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$$\check{T}^2 = \mathrm{Hom}(\mathbb{T}, T)^2 = \mathrm{Hom}(\mathbb{T}^2, T) \subset \mathrm{Map}(\mathbb{T}^2, T) =: L^2 T$$

Constructing the $\check{T}^2$-equivariant sheaf

For a finite $\mathbb{T}^2 \times T$-CW complex Y, define a holomorphic sheaf $\mathcal{H}_{\mathbb{T}^2\times T}(Y)$ on $\mathbb{H} \times \mathfrak{t}_{\mathbb{C}}$ whose value on an open subset U is

$$H_{\mathbb{T}^2\times T}(Y) \otimes_{H_{\mathbb{T}^2\times T}} \mathcal{O}_{\mathbb{H}\times\mathfrak{t}_{\mathbb{C}}}(U)$$

where the tensor product is over restriction of polynomials to $\mathbb{H} \times \mathfrak{t}_{\mathbb{C}} \subset \mathbb{C}^2 \times \mathfrak{t}_{\mathbb{C}}$.

Definition

Define the $\check{T}^2$-equivariant sheaf of $\mathcal{O}_{\mathbb{H}\times\mathfrak{t}_{\mathbb{C}}}$-algebras

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This is a $\check{T}^2$-equivariant sheaf of $\mathcal{O}_{\mathbb{H} \times \mathfrak{t}_{\mathbb{C}}}$-algebras, via the action of $\check{T}^2 \subset L^2 T$ on $L^2 X = \mathrm{Map}(\mathbb{T}^2, X)$.

The pushforward to $E_{T,\tau}$

Let $i_\tau : \mathfrak{t}_\mathbb{C} \hookrightarrow \mathbb{H} \times \mathfrak{t}_\mathbb{C}$ be the inclusion over τ, and let

$$\zeta : \mathfrak{t}_\mathbb{C} \twoheadrightarrow \mathfrak{t}_\mathbb{C}/(\check{T}\tau \times \check{T}) = E_{T,\tau}$$

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The restriction $i_\tau^* \mathcal{H}_{L^2 T}(L^2 X)$ to the fiber over τ is a $\check{T}^2$-equivariant sheaf of $\mathcal{O}_{\mathfrak{t}_\mathbb{C}}$-algebras.

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Theorem (S.)

There is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{O}_{E_{T,\tau}}$-algebras

$$\mathcal{E}\ell\ell_{T,\tau}(X) \cong \left(\zeta_* i_\tau^* \mathcal{H}_{L^2 T}(L^2 X)\right)^{\check{T}^2}$$

natural in X.

The K-theory of free loop space LX

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Definition

Define the $\check{T}$-equivariant sheaf of $\mathcal{O}_{\mathbb{D}^\times \times T_\mathbb{C}}$-algebras

$$\mathcal{K}_{LT}(LX) := \varprojlim_{Y \subset LX \text{ finite}} \mathcal{K}_{\mathbb{T} \times T}(Y).$$

The pushforward to $C_{T,q}$

Let $i_q : T_{\mathbb{C}} \hookrightarrow \mathbb{D}^{\times} \times T_{\mathbb{C}}$ be the inclusion over q, and let

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Theorem (S.)

Let $q = e^{2\pi i\tau}$. There is an isomorphism of $\mathcal{O}_{E_{T,\tau}}$-algebras

$$\mathcal{E}\ell\ell^0_{T,\tau}(X) \cong \left(\psi_* i_q^* \mathcal{K}_{LT}(LX)\right)^{\check{T}}$$

natural in X.

(Kitchloo, 2014) has constructed a cohomology theory ${}^k\mathcal{K}_{LT}(LX)$ out of level k loop group representations, which maps into $\mathcal{K}_{LT}(LX) \otimes \mathcal{L}^k$, and therefore into $\mathcal{E}\ell\ell_T(X) \otimes \mathcal{L}^k$, where $\mathcal{L}$ is the Looijenga line bundle.