A construction of complex analytic elliptic cohomology from double free loop spaces

Matthew Spong

Abstract

We construct a complex analytic version of an equivariant cohomology theory which appeared in a recent paper of Rezk, and which is roughly modeled on the Borel-equivariant cohomology of the double free loop space. The construction is defined on finite, torus-equivariant CW complexes and takes values in coherent holomorphic sheaves over the moduli stack of complex elliptic curves. Our methods involve an inverse limit construction over all finite dimensional subcomplexes of the double free loop space, following an analogous construction of Kitchloo for single free loop spaces. We show that, for any given complex elliptic curve $\mathcal{C}$, the fiber of our construction over $\mathcal{C}$ is isomorphic to Grojnowski's equivariant elliptic cohomology theory associated to $\mathcal{C}$.

In Section 5 of [10], Rezk considered an equivariant cohomology theory $E_T^*(X)$ modeled on the Borel-equivariant cohomology of a certain subspace L^2X^{gh} of the double free loop space

$$L^2X = \mathrm{Map}(S^1 \times S^1, X)$$

of a T-CW complex X, where T is a compact torus.[1] He conjectured that a complex analytic version of his construction would serve as a model for Grojnowski's equivariant elliptic cohomology theory [7]. However, tensoring $E_T^*(X)$ with holomorphic coefficients does not behave well, because $E_T^*(X)$ is often non-Noetherian, even when X is a T-orbit. In this article, we solve this problem by applying an idea that appeared in [8] in the context of single free loop spaces. Namely, the idea is to replace the cohomology of L^2X^{gh} with the inverse limit over all finite subcomplexes of L^2X, making sure to tensor with holomorphic coefficients before applying the limit. Tensoring in this fashion behaves well because the cohomology ring of a finite CW-complex is finitely generated. From this, we produce a complex analytic version $\mathcal{E}_T^*(X)$ of $E_T^*(X)$, which is a cohomology theory in X, and which takes values in coherent holomorphic sheaves over the moduli stack of complex elliptic curves. Furthermore, we show that the fiber of $\mathcal{E}_T^*$ over any given elliptic curve $\mathcal{C}$ is isomorphic to Grojnowski's construction associated to $\mathcal{C}$.

We now give an outline of the plan of the paper and mention the main results. In Section 1 we introduce the moduli stack $\mathcal{M}$ of complex elliptic curves, modeled as an equivariant space, and some other basic objects. In Section 2, we introduce Borel-equivariant cohomology and some of its important properties, and we summarise Grojnowski's construction in Section 3. In Section 4, we introduce the equivariant space E_T which will be the base space of the equivariant sheaf $\mathcal{E}_T^*(X)$. We show in Section 5 that, given a finite T-CW complex X, there exists an open cover of

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[1] The superscript gh stands for ghost maps. A ghost map is a map whose image is contained in a single T-orbit. Restricting to this space ensures that Rezk's functor is a cohomology theory in X.

E_T which is adapted to X in a certain sense. The open cover will allow us to obtain some useful fixed-point results in Section 6, after we introduce the inverse limit construction in Definition 6.2. In Section 7 we use the fixed-point results of the previous section to show that the inverse limit construction has a computable form (see Theorem 7.2). Then, in Definition 7.5 we give the construction of the equivariant sheaf $\mathcal{E}_T^*(X)$, whose value when $X = \text{pt}$ and $T = 1$ is the graded ring of weakly holomorphic modular forms (see Example 7.8). In Section 8, we show that $\mathcal{E}_T^*$ inherits a suspension isomorphism from ordinary cohomology, and that consequently $X \mapsto \mathcal{E}_T^*(X)$ is a cohomology theory in X with values in coherent holomorphic sheaves (Corollary 8.5). Section 9 is a calculation of the equality $\mathcal{E}_T^*(T/H) = \mathcal{E}_H^*(\text{pt})$, which allows one to compute $\mathcal{E}_T^*(X)$ for any X using the Mayer-Vietoris sequence. Finally, in Section 10, we give a local description of the fiber of $\mathcal{E}_T^*(X)$ over any given elliptic curve $\mathcal{C}$. In other words, we fix a curve $\mathcal{C}$ and an open cover adapted to some fixed X, we take the restriction of $\mathcal{E}_T^*(X)$ to $\mathcal{C} \in \mathcal{M}$, and we compute it as a collection of sheaves indexed by the elements of the open cover, along with some gluing maps. It turns out that this is exactly the T-equivariant elliptic cohomology of X, as constructed by Grojnowski in [7] for the elliptic curve $\mathcal{C}$. This is Corollary 10.8.

Conventions 0.1. For a group G acting on a space X, we use $g \cdot x$ to denote the action of $g \in G$ on $x \in X$, and we use gg' to denote the group product of $g, g' \in G$. If X and Y are topological spaces, then the set of continuous maps $\text{Map}(X, Y)$ is regarded as a space with the compact-open topology. Let A be an abelian group, let H be an arbitrary group, and let A act on H. Our convention for the group law of the semidirect product $A \ltimes H$ is

$$(a', h')(a, h) = (a'a, a^{-1} \cdot h'h).$$

The tensor product of two $\mathbb{Z}$-modules is over $\mathbb{Z}$, unless otherwise specified. All rings are assumed to have a multiplicative identity. For (not necessarily square) matrices A, m, and t, we use expressions such as Am, mt, and mA to mean matrix multiplication. So, for example, if $m = (m_1, m_2)$ and $t = (t_1, t_2)$ are vectors, then mt means the dot product, where the transpose of a vector should be understood whenever it is necessary to make sense of an expression.

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1. Elliptic curves over $\mathbb{C}$ and other basic objects

In this section, we list some well known facts concerning the classification of elliptic curves over $\mathbb{C}$, drawn from the short summary appearing in Section 2 of [10]. We also introduce some other basic objects, including the base space $E_{T,t}$ of Grojnowski's construction [7].

Remark 1.1. Consider the subspace

$$\mathcal{X} := \{(t_1, t_2) \in \mathbb{C}^2 \mid \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C}\} \subset \mathbb{C}^2.$$

An element $t = (t_1, t_2) \in \mathcal{X}$ defines a lattice

$$\Lambda_t := \mathbb{Z}t_1 + \mathbb{Z}t_2 \subset \mathbb{C}.$$

It is easily verified that $\mathcal{X}$ is preserved under left multiplication by $\mathrm{GL}_2(\mathbb{Z})$, and that $\Lambda_t = \Lambda_{t'}$ if and only if there is a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ such that $At = t'$.

Definition 1.2. An *elliptic curve over* $\mathbb{C}$ is a complex manifold

$$E_t := \mathbb{C}/\Lambda_t,$$

along with the quotient group structure induced by the additive group $\mathbb{C}$. A *map of elliptic curves* $E_t \to E_{t'}$ is induced by multiplication by a nonzero complex number λ satisfying $\lambda\Lambda_t \subset \Lambda_{t'}$. Such a map is an isomorphism if and only if $\lambda\Lambda_t = \Lambda_{t'}$.

Remark 1.3. Two elliptic curves E_t and $E_{t'}$ are equal if and only if there exists a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ such that $At = t'$, and isomorphisms $E_t \cong E_{t'}$ correspond bijectively to pairs $(\lambda, A) \in \mathrm{GL}_2(\mathbb{Z})$ such that $\lambda At = t'$. Therefore, elliptic curves over $\mathbb{C}$ are classified by the action of $\mathbb{C}^\times \times \mathrm{GL}_2(\mathbb{Z})$ on $\mathcal{X}$ given by $(\lambda, A) \cdot t = \lambda At$. Alternatively, they are also classified by the action of the subgroup

$$\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z}) \subset \mathbb{C}^\times \times \mathrm{GL}_2(\mathbb{Z})$$

on the subspace

$$\mathcal{X}^+ = \{(t_1, t_2) \in \mathcal{X} \,|\, \mathrm{Im}(t_1/t_2) > 0\} \subset \mathcal{X},$$

which is easily seen to inherit such an action. In this paper, a *sheaf on the moduli stack*

$$\mathcal{M} := (\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z}))\backslash\!\backslash\mathcal{X}^+$$

of complex elliptic curves means a $\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z})$-equivariant sheaf on $\mathcal{X}^+$.

Remark 1.4. In this paper, we write $\mathbb{T}$ for the additive circle $\mathbb{R}/\mathbb{Z}$. For a compact abelian Lie group T, define the cocharacter lattice of T as

$$\check{T} := \mathrm{Hom}(\mathbb{T}, T).$$

The canonical isomorphism $\check{T} \otimes \mathbb{T} \cong T$ then provides us with additive coordinates for T. We identify the Lie algebra $\mathfrak{t}$ of T with $\check{T} \otimes \mathbb{R}$ and the exponential Lie map with the map

$$\pi_T : \check{T} \otimes \mathbb{R} \longrightarrow \check{T} \otimes \mathbb{T}$$

induced by $\pi : \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$. The kernel of π_T is $\check{T} \otimes \mathbb{Z} \cong \check{T}$. Let $\hat{T}$ denote the character group $\mathrm{Hom}(T, \mathbb{T})$ of T, and define the complex manifolds

$$\mathfrak{t}_{\mathbb{C}} := \mathrm{Hom}(\hat{T}, \mathbb{C}) \quad \text{and} \quad E_{T,t} := \mathrm{Hom}(\hat{T}, E_t).$$

The map $\mathbb{C} \twoheadrightarrow E_t$ induces a complex analytic map

$$\zeta_{T,t} : \mathfrak{t}_{\mathbb{C}} \twoheadrightarrow E_{T,t}.$$

If T is a torus, then there is a canonical isomorphism $\check{T} \cong \mathrm{Hom}(\hat{T}, \mathbb{T})$ given by the perfect pairing

$$\begin{array}{rcl} \hat{T} \times \check{T} & \to & \mathbb{Z} \\ (\mu, m) & \mapsto & \mu \circ m. \end{array}$$

We therefore have canonical isomorphisms

$$\mathfrak{t}_{\mathbb{C}} \cong \check{T} \otimes \mathbb{C} \quad \text{and} \quad E_{T,t} \cong \check{T} \otimes E_t.$$

Definition 1.5. A T-CW complex X is a union

$$\bigcup_{n\in\mathbb{N}} X^n$$

of T-subspaces X^n such that

(i) X^0 is a disjoint union of orbits T/H, where $H \subset T$ is a closed subgroup; and

(ii) X^{n+1} is obtained from X^n by attaching T-cells $T/H \times D^{n+1}$ along T-equivariant attaching maps $T/H \times S^n \to X^n$, where T acts trivially on D^{n+1}.

A *finite T-CW complex* is a T-CW complex which is a union of finitely many T-cells. A *pointed T-CW complex* is a T-CW complex along with a distinguished T-fixed basepoint in the 0-skeleton of X. Given a T-CW complex X, we write X_+ for the pointed T-CW complex which is the disjoint union of X and the basepoint $* = T/T$. A map $f : X \to Y$ of (pointed) T-CW complexes is a T-equivariant map such that $f(X^n) \subset Y^n$ for all n (and preserving the basepoint).

Example 1.6. Let T be a rank one torus and let $\lambda \in \hat{T}$ be an irreducible character of T. The *representation sphere associated to* λ is the one-point compactification S_λ of the one dimensional complex representation $\mathbb{C}_\lambda$ associated to λ. This may be equipped with the structure of a finite T-CW complex where

$$X^0 = T/T \times \{\infty\} \amalg T/T \times \{0\},$$

and $X^1 = T \times D^1$, with T-equivariant attaching map $T \times S^0 \to X^0$ given by sending one end of D^1 to $\{0\}$ and the other end to $\{\infty\}$. An element $z \in T$ acts by multiplication by $\lambda(z)$ on the left factor of X^1 and trivially on the right factor.

2. Some properties of Borel-equivariant cohomology

We now introduce Borel-equivariant cohomology, a fundamental ingredient of our construction, and state several of its properties which will be useful to us. We first note that, for any topological group G, one obtains by the Milnor construction a contractible space EG with a free right action of G. For a G-space X, the Borel construction $EG\times_G X$ of X is the topological quotient of $EG\times X$ by the equivalence relation $(e\cdot g, x) \sim (e, g\cdot x)$. The Borel-equivariant cohomology $H_G^*(X)$ of X is then defined to be the ordinary cohomology of $EG\times_G X$, understood as a graded-commutative ring with the cup product. In this paper, Borel-equivariant cohomology has complex coefficients, unless stated otherwise. We will also write H_G^* for $H_G^*(\mathrm{pt})$, sometimes dropping the asterisk from our notation.

We return to the case where G is a compact torus T. Since the unique map from X to a point induces a map $H_T \to H_T^*(X)$ of graded rings, $H_T^*(X)$ is naturally equipped with the structure of a graded H_T-algebra. Our reference for Borel-equivariant ordinary cohomology is [2].

Proposition 2.1. *There is an isomorphism of graded rings*

$$H_T^*(T/H) \cong H_H.$$

Proof. Since H acts freely on ET, the space ET is a model for EH. Therefore,

$$ET \times_T T/H \cong ET/H$$

is a model for BH. □

A proof of the following result may be found at Proposition 2.3.4. in [3].

Proposition 2.2. *Let X be a finite T-CW complex such that $H \subset T$ acts trivially. There is an isomorphism of graded H_T-algebras*

$$H_K^*(X) \otimes_{H_K} H_T \cong H_T^*(X)$$

natural in X, and induced by $h^* \cup j^*$.

Remark 2.3. Let $T \twoheadrightarrow K \twoheadrightarrow G$ be a composition of surjective maps of compact abelian groups, where T is a torus. If X is a finite G-CW complex, then there is a commutative diagram

$$\begin{array}{ccc} ET \times_T X & \xrightarrow{j} & BT \\ \downarrow h & & \downarrow p \\ EK \times_K X & \xrightarrow{g} & BK \\ \downarrow f & & \downarrow \\ EG \times_G X & \longrightarrow & BG \end{array} \tag{1}$$

where both squares are pullback diagrams. By Proposition 2.2, we have an induced diagram of isomorphisms of graded H_T-algebras

$$\begin{array}{ccc} H_G(X) \otimes_{H_G} H_K \otimes_{H_K} H_T & \xrightarrow{f^* \cup g^* \otimes \mathrm{id}} & H_K(X) \otimes_{H_K} H_T \\ \downarrow & & \downarrow h^* \cup j^* \\ H_G(X) \otimes_{H_G} H_T & \xrightarrow{(f \circ h)^* \cup j^*} & H_T(X) \end{array} \tag{2}$$

where the left vertical map is the canonical map induced by $p^* : H_K \to H_T$.

Lemma 2.4. *The diagram (2) commutes.*

Proof. It suffices to show that diagram (2) commutes for an element of the form $a \otimes b \otimes c$. We have

$$\begin{array}{ccc} a \otimes b \otimes c & \longmapsto & (f^* a \cup g^* b) \otimes c \\ \downarrow & & \downarrow \\ a \otimes (p^* b \cup c) & \longmapsto & (f \circ h)^* a \cup j^*(p^* b \cup c) = h^*(f^* a \cup g^* b) \cup j^* c, \end{array}$$

where equality holds since

$$\begin{aligned} (f \circ h)^* a \cup j^*(p^* b \cup c) &= (f \circ h)^* a \cup (p \circ j)^* b \cup j^* c \\ &= (f \circ h)^* a \cup (g \circ h)^* b \cup j^* c \\ &= h^* f^* a \cup h^* g^* b \cup j^* c \\ &= h^*(f^* a \cup g^* b) \cup j^* c. \end{aligned}$$

□

Remark 2.5. The Chern-Weil isomorphism

$$\mathrm{Sym}^*(\mathfrak{t}_{\mathbb{C}}^*) \cong H^*(BT; \mathbb{C})$$

enables us to identify an element in H_T with a polynomial function on $\mathfrak{t}_{\mathbb{C}}$. The map is produced as follows. Note that since T is a torus, there is an identification

$$\hat{T} \otimes \mathbb{C} \cong \mathrm{Hom}(\check{T}, \mathbb{Z}) \otimes \mathbb{C} \cong \mathfrak{t}_{\mathbb{C}}^*.$$

Let $\mathbb{C}_\lambda$ be the representation corresponding to an irreducible character $\lambda \in \hat{T}$. The map

$$\lambda \mapsto c_1(ET \times_T \mathbb{C}_\lambda)$$

induces an isomorphism $\hat{T} \cong H^2(BT;\mathbb{Z})$, where c_1 denotes the first Chern class. Tensoring this map with $\mathbb{C}$ and extending by the symmetric product yields the isomorphism $S(\mathfrak{t}^*_{\mathbb{C}}) \cong H^*(BT;\mathbb{C})$. See Proposition 2.6 in [12] for details.

Definition 2.6. Let T be a torus and let X be a finite T-CW complex. We have reserved the notation $\mathcal{H}^*_T(X)$ for the holomorphic sheaf associated to $H^*_T(X)$, whose value on an analytic open set $U \subset \mathfrak{t}_{\mathbb{C}}$ is

$$H^*_T(X)_U := H^*_T(X) \otimes_{H_T} \mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(U)$$

with restriction maps induced by those of $\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}$. It follows from Propositions 2.8 and 2.10 in [12] that this is a sheaf, rather than just a presheaf. We write $\mathcal{H}^*_T(X)_V$ for the restriction of $\mathcal{H}^*_T(X)$ along the inclusion of any subset $V \subset \mathfrak{t}_{\mathbb{C}}$.

Proposition 2.7. *Let*

$$p : T \longrightarrow T/H$$

be a surjective map of compact abelian groups, with kernel H. The natural map

$$p^* \mathcal{H}^*_{T/H}(X) \longrightarrow \mathcal{H}^*_T(X)$$

is an isomorphism of $\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}$-algebras.

Proof. This follows immediately from Theorem 2.2 by extending to holomorphic sheaves. □

Definition 2.8. Let $x \in \mathfrak{t}_{\mathbb{C}}$. The inclusion of a closed subgroup $H \subset T$ induces an inclusion of complex Lie algebras

$$\mathrm{Lie}(H)_{\mathbb{C}} := \mathrm{Hom}(\hat{H}, \mathbb{C}) \hookrightarrow \mathrm{Hom}(\hat{T}, \mathbb{C}) =: \mathfrak{t}_{\mathbb{C}}.$$

Define the intersection

$$T(x) = \bigcap_{x \in \mathrm{Lie}(H)_{\mathbb{C}}} H$$

of closed subgroups $H \subset T$. For a finite T-CW complex X, denote by X^x the subspace of points fixed by $T(x)$.

We now state a well known fixed-point theorem for Borel-equivariant cohomology. A proof may be found at Theorem 2.2.18 in [13].

Theorem 2.9. *Let $x \in \mathfrak{t}_{\mathbb{C}}$ and X be a finite T-CW complex. The restriction along $X^x \hookrightarrow X$ induces an isomorphism*

$$\mathcal{H}^*_T(X)_x \longrightarrow \mathcal{H}^*_T(X^x)_x$$

of $\mathcal{O}_{\mathfrak{t}_{\mathbb{C}},x}$-algebras.

3. Grojnowski's equivariant elliptic cohomology

There are already many accounts of the construction of Grojnowski's equivariant elliptic cohomology theory (e.g. [1], [3], [5], [6], [7], and [11]). Nonetheless, we will sketch a brief description here because it is important for our main results. In this section, we fix an elliptic curve $E_t = \mathbb{C}/\Lambda_t$.

Definition 3.1. Let $a \in E_{T,t}$. Define $T(a)$ as the intersection

$$T(a) = \bigcap_{a \in E_{H,t}} H$$

of closed subgroups $H \subset T$. For a T-CW complex X, denote by X^a the subspace of points fixed by $T(a)$.

REMARK 3.2. If $\mathcal{S}$ is a finite set of closed subgroups of T, we can define an ordering on the points of $E_{T,t}$ by saying that $a \leqslant_{\mathcal{S}} b$ if $b \in E_{H,t}$ implies $a \in E_{H,t}$, for any $H \in \mathcal{S}$. If $\mathcal{S}$ is understood, then we just write $\leqslant$ for $\leqslant_{\mathcal{S}}$.

NOTATION 3.3. If X is a finite T-CW complex, let $\mathcal{S}(X)$ be the finite set of isotropy subgroups of X. If $f : X \to Y$ is a map of finite T-CW complexes, let $\mathcal{S}(f)$ be the finite set of isotropy subgroups which occur in either X or Y. An open set U in $E_{T,t}$ is *small* if $\zeta_{T,t}^{-1}(U)$ is a disjoint union of connected components V such that $V \cong U$ via $\zeta_{T,t}$.

DEFINITION 3.4. Let $\mathcal{S}$ be a finite set of closed subgroups of T. An open cover $\mathcal{U} = \{U_a\}$ of $E_{T,t}$ indexed by the points of $E_{T,t}$ is said to be *adapted to* $\mathcal{S}$ if it has the following properties:

(i) $a \in U_a$, and U_a is small.
(ii) If $U_a \cap U_b \neq \emptyset$, then either $a \leqslant_{\mathcal{S}} b$ or $b \leqslant_{\mathcal{S}} a$.
(iii) If $a \leqslant_{\mathcal{S}} b$, and for some $H \in \mathcal{S}$, we have $a \in E_{H,t}$ but $b \notin E_{H,t}$, then $U_b \cap E_{H,t} = \emptyset$.
(iv) Let a and b lie in $E_{H,t}$ for some $H \in \mathcal{S}$. If $U_a \cap U_b \neq \emptyset$, then a and b belong to the same connected component of $E_{H,t}$.

The following result is Theorem 2.2.8 in [3].

LEMMA 3.5. *For any finite set $\mathcal{S}$ of subgroups of T, there exists an open cover $\mathcal{U}$ of $E_{T,t}$ adapted to $\mathcal{S}$. Any refinement of $\mathcal{U}$ is also adapted to $\mathcal{S}$.*

NOTATION 3.6. For $a \in E_{T,t}$ let

$$t_a : E_{T,t} \longrightarrow E_{T,t}$$

denote translation by a.

REMARK 3.7. Let X be a finite T-CW complex and let $\mathcal{U}$ be a cover of $E_{T,t}$ which is adapted to $\mathcal{S}(X)$. Let $x \in \zeta_{T,t}^{-1}(a)$, and let V_x be the component of $\zeta_{T,t}^{-1}(U_a)$ containing x. Let $V \subset V_x$ and $U \subset U_a$ be open subsets such that $V \cong U$ via $\zeta_{T,t}$. Since $U_a \in \mathcal{U}$ is small by the first property of an adapted cover, the map $\zeta_{T,t}$ induces an isomorphism of complex analytic spaces $V - x \cong U - a$. We may therefore consider the composite ring map

$$H_T \hookrightarrow \mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V - x) \cong \mathcal{O}_{E_{T,t}}(U - a). \tag{3}$$

DEFINITION 3.8. Let X be a finite T-CW complex. For each $U_a \in \mathcal{U}(X)$, define a sheaf $\mathcal{G}^*_{T,t}(X)_{U_a}$ of $\mathbb{Z}/2\mathbb{Z}$-graded O_{U_a}-algebras which takes the value

$$H_T(X^a) \otimes_{H_T} \mathcal{O}_{E_{T,t}}(U - a),$$

on $U \subset U_a$ open, with restriction maps given by restriction of holomorphic functions. The tensor product is defined over (3), and the $\mathcal{O}_{U_a}$-algebra structure is given by multiplication by $t_a^* f$ for $f \in \mathcal{O}_{U_a}(U)$. The grading is induced by the odd and even grading on the cohomology ring.

REMARK 3.9. For a finite T-CW complex X, we have defined a sheaf on each patch U_a of a cover $\mathcal{U}$ adapted to $\mathcal{S}(X)$. The next task is to glue the local sheaves together on nonempty intersections $U_a \cap U_b$ in a compatible way. To do this, we need to define gluing maps

$$\phi_{b,a} : \mathcal{G}_{T,t}(X)_{U_a}|_{U_a \cap U_b} \cong \mathcal{G}_{T,t}(X)_{U_b}|_{U_a \cap U_b}$$

for each such intersection, such that the cocycle condition $\phi_{c,b} \circ \phi_{b,a} = \phi_{c,a}$ is satisfied whenever $U_a \cap U_b \cap U_c \neq \emptyset$.

Note that we have either $X^b \subset X^a$ or $X^a \subset X^b$ whenever $U_a \cap U_b \neq \emptyset$, by the second property of an adapted cover.

Theorem 3.10. *Let X be a finite T-CW complex, and let $\mathcal{U}$ be a cover adapted to $\mathcal{S}(X)$. Let $a \leqslant b$ be points in $E_{T,t}$ and assume $U \subset U_a \cap U_b$ is an open subset. By the second property of an adapted cover, we may assume that $X^b \subset X^a$, with inclusion map $i_{b,a}$. The map*

$$i_{b,a}^* \otimes \mathrm{id} : H_T(X^a) \otimes_{H_T} \mathcal{O}_{E_{T,t}}(U-a) \to H_T(X^b) \otimes_{H_T} \mathcal{O}_{E_{T,t}}(U-a)$$

induced by restriction along $i_{b,a}$ is an isomorphism of $O_{E_{T,t}}(U)$-modules.

Proof. See the proof of Theorem 2.3.3 in [3]. □

Remark 3.11. Let $H = \langle T(a), T(b) \rangle$ and let $U \subset U_a \cap U_b$ be an open subset. There is a natural isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{O}(U)$-algebras given on $U \subset U_a$ by the composite

$$\begin{aligned} H_T(X^a) \otimes_{H_T} \mathcal{O}_{E_{T,t}}(U-a) \quad &\xrightarrow{i_{b,a}^* \otimes \mathrm{id}} \quad H_T(X^b) \otimes_{H_T} \mathcal{O}_{E_{T,t}}(U-a) \\ &\longrightarrow \quad H_{T/H}(X^b) \otimes_{H_{T/H}} \mathcal{O}_{E_{T,t}}(U-a) \\ &\xrightarrow{\mathrm{id} \otimes t_{b-a}^*} \quad H_{T/H}(X^b) \otimes_{H_{T/H}} \mathcal{O}_{E_{T,t}}(U-b) \\ &\longrightarrow \quad H_T(X^b) \otimes_{H_T} \mathcal{O}_{E_{T,t}}(U-b), \end{aligned} \tag{4}$$

where the second and final maps are the isomorphism of Proposition 2.2. Denote the composite by $\phi_{b,a}$.

Remark 3.12. We take the opportunity to correct a small error in the literature. In Definition 2.3.5. in [3], the gluing maps $\phi_{b,a}$ are defined using the map

$$H_{T/T(b)}(X^b) \otimes_{H_{T/T(b)}} \mathcal{O}(U-a) \xrightarrow{1 \otimes t_{b-a}^*} H_{T/T(b)}(X^b) \otimes_{H_{T/T(b)}} \mathcal{O}(U-b). \tag{5}$$

However, t_{b-a}^* does not always preserve the $H_{T/T(b)}$-algebra structure, because $b-a$ is not always contained in $E_{T(b),t}$. To see this, take X equal to a point, so that $\mathcal{S}(X) = \{T\}$, and set $a = [t_1/2]$ and $b = [0]$. Thus, $T(a) = \mathbb{Z}/2\mathbb{Z}$ and $T(b) = 1$, and in this case, $b-a$ is equal to $[-t_1/2]$ which is not in $E_{T(b),t} = 0$.

This is easily fixed by setting H equal to $\langle T(a), T(b) \rangle$ and using the change of groups map associated to $T \to T/H$. By definition of $T(a)$ and $T(b)$, we have that $b - a \in E_{H,t}$, and it follows that $t_{b-a}^* : \mathcal{O}(U-a) \to \mathcal{O}(U-b)$ is a map of $H_{T/H}$-algebras. So, we replace (5) with the map

$$H_{T/H}(X^b) \otimes_{H_{T/H}} \mathcal{O}(U-a) \xrightarrow{1 \otimes t_{b-a}^*} H_{T/H}(X^b) \otimes_{H_{T/H}} \mathcal{O}(U-b)$$

in our account of Grojnowski's construction. Note that X^b is fixed by H.

The following result is Proposition 2.3.7 in [3].

Proposition 3.13. *The collection of maps $\{\phi_{b,a}\}$ satisfies the cocycle condition*

$$\phi_{c,b} \circ \phi_{b,a} = \phi_{c,a}$$

whenever $U_a \cap U_b \cap U_c \neq \emptyset$.

Definition 3.14. We denote by $\mathcal{G}_{T,t}^*(X)$ the sheaf of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{O}_{E_{T,t}}$-algebras which is obtained by gluing together the sheaves $\mathcal{G}_{T,t}^*(X)_{U_a}$ via the maps $\phi_{b,a}$.

Remark 3.15. Up to isomorphism, the sheaf $\mathcal{G}_{T,t}^*(X)$ does not depend on the choice of $\mathcal{U}$ since any refinement of $\mathcal{U}$ is also adapted to $\mathcal{S}(X)$. More explicitly, given two covers $\mathcal{U}$ and $\mathcal{U}'$ adapted to $\mathcal{S}(X)$, one may take the common refinement $\mathcal{U}''$ and consider the theory defined using $\mathcal{U}''$. The resulting theory is then naturally isomorphic to those theories corresponding to $\mathcal{U}$ and $\mathcal{U}'$, since the maps induced by the refinement are isomorphisms on stalks.

The following is Theorem 2.3.8 in [3]. We reproduce the proof here as it is important for our main results.

Proposition 3.16. *The functor $X \mapsto \mathcal{G}^*_{T,t}(X)$ is a T-equivariant cohomology theory defined on finite T-CW complexes with values in coherent sheaves of $\mathbb{Z}/2$-graded $\mathcal{O}_{E_{T,t}}$-algebras.*

Proof. That $\mathcal{G}^*_{T,t}(X)$ is a coherent sheaf simply follows from the fact that X is a finite T-CW complex, and that $\mathcal{G}^*_{T,t}(X)$ may be computed locally using ordinary cellular cohomology. We show that the construction of $\mathcal{G}_{T,t}(X)$ is functorial in X. Let $f : X \to Y$ be a map of finite T-CW complexes and let $\mathcal{U}$ be a cover of $E_{T,t}$ which is adapted to $\mathcal{S}(f)$. For $a \in E_{T,t}$, the map f induces a map $f_a : X^a \to Y^a$ by restriction. This induces a map

$$f_a^* \otimes \mathrm{id} : H_T(Y^a) \otimes_{H_T} \mathcal{O}(U - a) \to H_T(X^a) \otimes_{H_T} \mathcal{O}(U - a)$$

for each $U \subset U_a$, which clearly commutes with the restriction maps of the sheaf. It is evident that the collection of such maps for all $a \in E_{T,t}$ glue well, and that identity maps and composition of maps are preserved, by the functoriality of Borel-equivariant cohomology and naturality of the isomorphism of Proposition 2.2.

Define the reduced theory on pointed finite T-CW complexes by setting

$$\widetilde{\mathcal{G}}^*_{T,t}(X, A) := \ker(i^* : \mathcal{G}^*_{T,t}(X/A, *) \to \mathcal{G}^*_{T,t}(\mathrm{pt}))$$

where $i : \mathrm{pt} \hookrightarrow X$ is the inclusion of the basepoint. By naturality of $\mathcal{G}^*_T$, a map $f : X \to Y$ of pointed complexes gives rise to a unique map $f^* : \widetilde{\mathcal{G}}^*_{T,t}(X, A) \to \widetilde{\mathcal{G}}^*_{T,t}(Y, B)$ on the corresponding kernels. This is functorial for the reasons already set out for single complexes.

Define a suspension isomorphism $\widetilde{\mathcal{G}}^{*+1}_{T,t}(S^1 \wedge X) \to \widetilde{\mathcal{G}}^*_{T,t}(X)$ by gluing the maps

$$\sigma_a \otimes \mathrm{id} : \widetilde{H}^{*+1}_T(S^1 \wedge X^a) \otimes_{H^*_T} \mathcal{O}(U - a) \to \widetilde{H}^*_T(X^a) \otimes_{H^*_T} \mathcal{O}(U - a),$$

where σ_a is the suspension isomorphism of Borel-equivariant cohomology. The maps $\sigma_a \otimes \mathrm{id}$ glue well since σ_a is natural.

Finally, the properties of exactness and additivity may be checked on stalks

$$\widetilde{\mathcal{G}}_{T,t}(X)_a = \widetilde{H}_T(X^a) \otimes_{H_T} \mathcal{O}_{E_{T,t},0}.$$

This is clear, since Borel-equivariant cohomology satisfies these properties, and tensoring with $\mathcal{O}_{E_{T,t},0} \cong \mathcal{O}_{\mathfrak{t}_{\mathbb{C}},0}$ is exact. □

4. The $\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z})$-equivariant complex manifold E_T

In this section, we work out the details of the picture sketched by Rezk in Section 2.12 of [10] (see also Etingof and Frenkel [4]). Namely, we construct a $\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z})$-equivariant complex manifold E_T as an equivariant fiber bundle over $\mathcal{X}^+$, such that the fiber over t is equal to $E_{T,t} = \check{T} \otimes E_t$. The manifold E_T will be the base space of the $\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z})$-equivariant sheaf $\mathcal{E}_T(X)$ that we construct in a later section.

Remark 4.1. Consider the semidirect product group

$$\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2$$

where $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by left multiplication. The group operation is given by

$$(A', t')(A, t) = (A'A, A^{-1}t' + t),$$

so that the inverse of (A, t) is $(A^{-1}, -At)$. We may think of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2$ as the group of orientation-preserving diffeomorphisms

$$\begin{array}{rcl} (A,t) : \mathbb{T}^2 & \longrightarrow & \mathbb{T}^2 \\ s & \longmapsto & As + t. \end{array}$$

Let L^2T be the topological group of smooth maps $\mathbb{T}^2 \to T$, with group multiplication defined pointwise. A diffeomorphism (A, s) acts on a loop $\gamma \in L^2T$ from the left by

$$(A, t) \cdot \gamma(s) = \gamma(A^{-1}s - At).$$

Definition 4.2. Following [10], define the *extended double loop group of T* as the semidirect product

$$\mathcal{G} := (\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2) \ltimes L^2T$$

with group operation

$$(A', t', \gamma'(s))(A, t, \gamma(s)) = (A'A, A^{-1}t' + t, \gamma'(As + t) + \gamma(s)). \tag{6}$$

One may think of an element $(A, t, \gamma) \in \mathcal{G}$ as the automorphism

$$\begin{array}{ccc} \mathbb{T}^2 \times T & \xrightarrow{\phi} & \mathbb{T}^2 \times T \\ \downarrow & & \downarrow \\ \mathbb{T}^2 & \xrightarrow{(A,t)} & \mathbb{T}^2 \end{array}$$

covering the diffeomorphism (A, t) of $\mathbb{T}^2$, where $\phi(r, s)$ is equal to $\gamma(r) + s$. It is easily verified that the inverse of $(A, t, \gamma(s))$ is equal to $(A^{-1}, -At, -\gamma(A^{-1}s - At))$.

Remark 4.3. For a finite T-CW complex X, the extended double loop group $\mathcal{G}$ acts on the double loop space

$$L^2X := \mathrm{Map}(\mathbb{T}^2, X)$$

by

$$(A, t, \gamma(s)) \cdot \gamma'(s) = \gamma(A^{-1}s - At) + \gamma'(A^{-1}s - At). \tag{7}$$

Definition 4.4. Consider the subgroup

$$\mathbb{T}^2 \times T \subset \mathbb{T}^2 \ltimes L^2T$$

where the translations $\mathbb{T}^2$ act trivially on the subgroup of constant loops $T \subset L^2T$. One sees that this is a maximal torus in $\mathcal{G}$ by noting that the intersection of $\mathrm{SL}_2(\mathbb{Z})$ with the identity component of $\mathcal{G}$ is trivial, and that a nonconstant loop in L^2T does not commute with $\mathbb{T}^2$. Let $N_{\mathcal{G}}(\mathbb{T}^2 \times T)$ be the normaliser of $\mathbb{T}^2 \times T$ in $\mathcal{G}$. The Weyl group associated to $\mathbb{T}^2 \times T \subset \mathcal{G}$ is defined as

$$W_{\mathcal{G}} = W_{\mathcal{G}}(\mathbb{T}^2 \times T) := N_{\mathcal{G}}(\mathbb{T}^2 \times T)/(\mathbb{T}^2 \times T).$$

Remark 4.5. In the following proposition, we will consider the subgroup

$$\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2 \subset \mathcal{G}$$

where $m \in \check{T}^2$ is identified with the loop $\gamma(s) = ms \in L^2T$ via

$$\check{T}^2 := \mathrm{Hom}(\mathbb{T}, T)^2 \cong \mathrm{Hom}(\mathbb{T}^2, T) \subset L^2T.$$

The group operation, induced by that of $\mathcal{G}$, is given by

$$(A', m')(A, m) = (A'A, m'A + m) \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2,$$

and the inverse of (A, m) is given by $(A^{-1}, -mA^{-1})$.

Proposition 4.6. *The subgroup $\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2 \subset \mathcal{G}$ is contained in $N_{\mathcal{G}}(\mathbb{T}^2 \times T)$, and the composite map*

$$\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2 \hookrightarrow N_{\mathcal{G}}(\mathbb{T}^2 \times T) \twoheadrightarrow W_{\mathcal{G}}(\mathbb{T}^2 \times T)$$

is an isomorphism.

Proof. Let $(A, m) \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2$. A straightforward calculation using (6) shows that

$$(A, 0, m)(1, r, t)(A^{-1}, 0, -mA^{-1}) = (1, Ar, t + mr) \in \mathbb{T}^2 \times T,$$

which proves the first assertion. For the second assertion, it suffices to define an inverse to the composite map of the proposition. Let g be an arbitrary element in $N_{\mathcal{G}}(\mathbb{T}^2 \times T)$ and let $[g]$ be its image in $W_{\mathcal{G}}$. By definition of $W_{\mathcal{G}}$, we may translate g by elements of $\mathbb{T}^2$, and act on g by constant loops, without changing $[g]$. Therefore, there exists $\gamma \in L^2T$ with $\gamma(0, 0) = 1$ such that

$$[g] = [(A, 0, \gamma)] \in W_{\mathcal{G}},$$

for some $A \in \mathrm{SL}_2(\mathbb{Z})$. We will now show that $\gamma \in \check{T}^2$, and finally that $[g] \mapsto (A, \gamma)$ is a well defined inverse to the composite map. Using (6) again, for any $(r, t) \in \mathbb{T}^2 \times T$, we have

$$(A, 0, \gamma(s))(r, t)(A, 0, -\gamma(s)) = (1, Ar, \gamma(r + A^{-1}s) + t - \gamma(A^{-1}s)) \in \mathbb{T}^2 \times T. \tag{8}$$

It follows that $\gamma(r + A^{-1}s) - \gamma(A^{-1}s)$ does not depend on s. Thus,

$$\gamma(r + A^{-1}s) - \gamma(A^{-1}s) = \gamma(r)$$

for all $r, s \in \mathbb{T}^2$, and setting $s = As'$ shows that $\gamma(r)+\gamma(s') = \gamma(r+s')$ for all $r, s' \in \mathbb{T}^2$. Therefore, γ is a group homomorphism, which means that it lies in $\check{T}^2$. The map $[g] \mapsto (A, \gamma)$ is well defined, since $g \in \mathbb{T}^2 \times T$ allows us to choose $A = 1$ and $\gamma = 1$, and is evidently a group homomorphism which is inverse to the composite map of the proposition. The completes the proof. □

Remark 4.7. It follows directly from equation (8) that the action of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2$ on $\mathbb{T}^2 \times T$ is given by

$$(A, m) \cdot (r, t) = (Ar, t + mr).$$

The induced action of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2 \subset \mathcal{G}$ on the complex Lie algebra $\mathbb{C}^2 \times \mathfrak{t}_{\mathbb{C}}$ is given by the same formula, in which case we write it as

$$(A, m) \cdot (t, x) = (At, x + mt).$$

Remark 4.8. Since $\mathrm{SL}_2(\mathbb{Z})$ preserves the subspace $\mathcal{X}^+ \subset \mathbb{C}^2$, the action of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2$ on $\mathbb{C}^2 \times \mathfrak{t}_{\mathbb{C}}$ preserves $\mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}$. As the action of $\check{T}^2$ is free and properly discontinuous, the quotient map

$$\zeta_T : \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}} \twoheadrightarrow \check{T}^2 \backslash (\mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}) =: E_T$$

is a complex analytic map and the quotient space is a complex manfiold. The residual action of $\mathrm{SL}_2(\mathbb{Z})$ descends to E_T, because $\check{T}^2$ is a normal subgroup. In Section 7, when we construct the equivariant sheaf $\mathcal{E}^*_T(X)$ over E_T, it will be necessary to consider the scalar action of $\mathbb{C}^\times$ on $\mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}$ given by $\lambda \cdot (t, x) = \lambda^2(t, x)$. This clearly commutes with the $\mathrm{SL}_2(\mathbb{Z})$-action, and we may view E_T as a $\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z})$-equivariant complex analytic space. We denote by $\mathcal{M}_T$ the quotient stack

$$\mathcal{M}_T := (\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z})) \backslash\!\backslash E_T$$

In this paper, a sheaf over $\mathcal{M}_T$ means a $\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z})$-equivariant sheaf on E_T.

Replace the scalar action on $\mathcal{X}^+$ in Remark 1.3 with its square. We view $\mathcal{M}_T$ as being equipped with the map $\mathcal{M}_T \to \mathcal{M}$ induced by $T \to \mathrm{pt}$. The fiber of the underlying fiber bundle $E_T \twoheadrightarrow \mathcal{X}^+$ over $t \in \mathcal{X}^+$ is given by

$$E_{T,t} = (\check{T}t_1 + \check{T}t_2)\backslash \mathfrak{t}_{\mathbb{C}} = (\Lambda_t \backslash \mathbb{C}) \otimes \check{T},$$

which the reader may recall is the base space of Grojnowski's construction. We denote by $\zeta_{T,t}$ the restriction of the map ζ_T to the fiber over $t \in \mathcal{X}^+$.

Remark 4.9. We can view $\mathcal{X}^+$ as a parameter space of complex structures on $\mathfrak{t} \times \mathfrak{t}$ and $T \times T$. Denote by ξ_T the map

$$\begin{array}{ccc} \mathcal{X}^+ \times \mathfrak{t} \times \mathfrak{t} & \longrightarrow & \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}} \\ (t_1, t_2, x_1, x_2) & \mapsto & (t_1, t_2, x_1 t_1 + x_2 t_2). \end{array}$$

The restriction of ξ_T to the fiber over $t \in \mathcal{X}^+$ is the isomorphism

$$\xi_{T,t} : \check{T} \otimes \mathbb{R}^2 \cong \check{T} \otimes \mathbb{C}$$

of the underlying real vector spaces induced by the complex linear structure $\mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C}$ on $\mathbb{R}^2$. Taking the quotient by $\check{T}^2$ induces a commutative diagram

$$\begin{array}{ccc} \mathcal{X}^+ \times \mathfrak{t} \times \mathfrak{t} & \xrightarrow{\xi_T} & \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}} \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \pi} & & \Big\downarrow{\scriptstyle \zeta_T} \\ \mathcal{X}^+ \times T \times T & \xrightarrow{\chi_T} & E_T \end{array} \tag{9}$$

of the underlying real manifolds. The restriction of χ_T to the fiber over $t \in \mathcal{X}^+$ is the isomorphism

$$\chi_{T,t} : \check{T} \otimes \mathbb{T}^2 \cong \check{T} \otimes E_t$$

of real Lie groups induced by the complex manifold structure

$$\mathbb{T}^2 \cong (\mathbb{R}t_1 + \mathbb{R}t_2)/(\mathbb{Z}t_1 + \mathbb{Z}t_2) = \mathbb{C}/\Lambda_t =: E_t.$$

5. An open cover of E_T adapted to X

In this section, we begin by defining, for a finite T-CW complex X, an open cover of the compact Lie group $T \times T$ adapted to X. We show that such a cover exists, and that it induces an open cover of the total space E_T via the isomorphism $\mathcal{X}^+ \times T \times T \cong E_T$. We also show that the restriction of the cover to $E_{T,t}$ is adapted to X in the sense of Definition 3.4. Finally, we establish some properties of the open cover which will be useful in later sections, and we give an example of a cover adapted to the representation sphere S_λ.

Remark 5.1 Ordering on $T \times T$. If $\mathcal{S}$ is a finite set of closed subgroups of T, we can define a relation on the points of $T \times T$ by saying that $(a_1, a_2) \leqslant_{\mathcal{S}} (b_1, b_2)$ if $(b_1, b_2) \in H \times H$ implies $(a_1, a_2) \in H \times H$, for any $H \in \mathcal{S}$. This relation is obviously reflexive and transitive, but not symmetric.

Notation 5.2. If X is a finite T-CW complex, let $\mathcal{S}(X)$ be the finite set of isotropy subgroups of X. If $f : X \to Y$ is a map of finite T-CW complexes, let $\mathcal{S}(f)$ be the finite set of isotropy subgroups which occur in either X or Y. An open set U in $T \times T$ is *small* if $\pi^{-1}(U)$ is a disjoint union of connected components V such that $\pi|_V : V \to U$ is a bijection for each V.

Definition 5.3 Open cover of $T \times T$. An open cover $\mathcal{U} = \{U_{a_1,a_2}\}$ of $T \times T$ indexed by the points of $T \times T$ is said to be *adapted to* $\mathcal{S}$ if it has the following properties:

(i) $(a_1, a_2) \in U_{a_1,a_2}$, and U_{a_1,a_2} is small.

(ii) If $U_{a_1,a_2} \cap U_{b_1,b_2} \neq \emptyset$, then either $(a_1, a_2) \leqslant_{\mathcal{S}} (b_1, b_2)$ or $(b_1, b_2) \leqslant_{\mathcal{S}} (a_1, a_2)$.

(iii) If $(a_1, a_2) \leqslant_{\mathcal{S}} (b_1, b_2)$, and for some $H \in \mathcal{S}$, we have $(a_1, a_2) \in H \times H$ but $(b_1, b_2) \notin H \times H$, then $U_{b_1,b_2} \cap H \times H = \emptyset$.

(iv) Let (a_1, a_2) and (b_1, b_2) lie in $H \times H$ for some $H \in \mathcal{S}$. If $U_{a_1,a_2} \cap U_{b_1,b_2} \neq \emptyset$, then (a_1, a_2) and (b_1, b_2) belong to the same connected component of $H \times H$.

If $\mathcal{S} = \mathcal{S}(X)$ and $\mathcal{U}$ is adapted to $\mathcal{S}$ then we say that $\mathcal{U}$ is adapted to X. If $\mathcal{S} = \mathcal{S}(f)$ and $\mathcal{U}$ is adapted to $\mathcal{S}$ then we say that $\mathcal{U}$ is adapted to f. If $\mathcal{S}$ is understood, then we just write $\leqslant$ for $\leqslant_{\mathcal{S}}$.

Our proof of the following result is based on the proof of Proposition 2.5 in [12].

Lemma 5.4. *For any finite set $\mathcal{S}$ of subgroups of T, there exists an open cover $\mathcal{U}$ of $\mathcal{E}_T$ adapted to $\mathcal{S}$. Any refinement of $\mathcal{U}$ is also adapted to $\mathcal{S}$.*

Proof. Since a compact abelian group has finitely many components, the set

$$\mathcal{S}^0 = \{D \subset T \times T \mid D \text{ is a component of } H \times H \text{ for some } H \in \mathcal{S}\}$$

is finite. Let d be the metric on $T \times T$ defined by

$$d((a_1, a_2), (b_1, b_2)) := \min\{d_{\mathfrak{t}\times\mathfrak{t}}((a_1, a_2) + m, (b_1, b_2) + m') \mid m, m' \in \check{T} \times \check{T}\}$$

where $d_{\mathfrak{t}\times\mathfrak{t}}$ denotes the Euclidean metric on $\mathfrak{t} \times \mathfrak{t}$. If $(a_1, a_2) \in D$ for all $D \in \mathcal{S}^0$, then define U_{a_1,a_2} to be an open ball centered at (a_1, a_2) with radius $r = \frac{1}{2}$. Otherwise, define U_{a_1,a_2} to be an open ball centered at (a_1, a_2), with radius

$$r = \frac{1}{2} \min\{D \mid (a_1, a_2) \notin D\},$$

where

$$d((a_1, a_2), D) = \min\{d((a_1, a_2), (b_1, b_2)) \mid (b_1, b_2) \in D\}.$$

The open cover $\mathcal{U}$ of $T \times T$ thus constructed clearly satisfies the first condition of an adapted cover.

Furthermore, if there exist distinct components $D, D' \in \mathcal{S}^0$ such that (a_1, a_2) is in D but not in D', and (b_1, b_2) is in D' but not in D, then $U_{a_1,a_2} \cap U_{b_1,b_2}$ is empty by construction. If D and D' correspond to distinct elements of $\mathcal{S}$, then (a_1, a_2) and (b_1, b_2) do not relate under the ordering, and the previous statement implies the contrapositive of the second condition. If D and D' correspond to the same element of $\mathcal{S}$, then it implies the contrapositive of the fourth condition.

The third condition holds simply because $(b_1, b_2) \notin H \times H$ always implies that $U_{b_1,b_2} \cap H \times H = \emptyset$, by construction. It is clear that any refinement of $\mathcal{U}$ will also satisfy all four conditions. □

Remark 5.5. Let $\mathcal{U}$ be a cover adapted to $\mathcal{S}(X)$. We will sometimes refer to $\mathcal{U}$ as being adapted to X, since $\mathcal{S}(X)$ is completely determined by X.

Lemma 5.6. *Let $\mathcal{U}$ be an open cover of $T \times T$ adapted to $\mathcal{S}$. If $(b_1, b_2) \in U_{a_1,a_2}$, then $(a_1, a_2) \leqslant (b_1, b_2)$.*

Proof. Suppose that $(b_1, b_2) \in U_{a_1,a_2}$ and $(a_1, a_2) \leqslant (b_1, b_2)$ does not hold. This implies two things. Firstly, by the second property of an adapted cover, we have $(b_1, b_2) \leqslant (a_1, a_2)$. Secondly, by definition of the relation, there must exist some $H \in \mathcal{S}$ such that $(b_1, b_2) \in H \times H$ and $(a_1, a_2) \notin H \times H$. Together, the two statements imply that $U_{a_1,a_2} \cap H \times H = \emptyset$, by the third property of an adapted cover. This contradicts the assumption that $(b_1, b_2) \in U_{a_1,a_2}$, since $(b_1, b_2) \in H \times H$. □

Definition 5.7. Let $\mathcal{U} = \{U_{a_1,a_2}\}$ be an open cover adapted to $\mathcal{S}$. We denote by $V_{x_1,x_2} \subset \mathfrak{t} \times \mathfrak{t}$ the open subset which is the component of $\pi^{-1}(U_{\pi(x_1,x_2)})$ containing (x_1, x_2).

The following lemma is a strengthening of the fourth property of an adapted cover, which we will need in the next section.

Lemma 5.8. *Let $\mathcal{U}$ be a cover of $T \times T$ adapted to $\mathcal{S}$. Let $(a_1, a_2), (b_1, b_2) \in T \times T$ with open neighbourhoods $U_{a_1,a_2}, U_{b_1,b_2} \in \mathcal{U}$, and let $(x_1, x_2), (y_1, y_2) \in \mathfrak{t} \times \mathfrak{t}$ such that $\pi(x_i) = a_i$ and $\pi(y_i) = b_i$. Let $H \in \mathcal{S}$ and suppose $(a_1, a_2), (b_1, b_2) \in H \times H$. If*

$$V_{x_1,x_2} \cap V_{y_1,y_2} \neq \emptyset,$$

then (x_1, x_2) and (y_1, y_2) lie in the same component of $\pi^{-1}(H \times H)$.

Proof. Since $V_{x_1,x_2} \cap V_{y_1,y_2} \neq \emptyset$, we have $U_{a_1,a_2} \cap U_{b_1,b_2} \neq \emptyset$. Therefore, (a_1, a_2) and (b_1, b_2) lie in the same component D of $H \times H$, by the fourth property of an adapted cover, so that $(x_1, x_2), (y_1, y_2) \in \pi^{-1}(D)$. We have

$$\begin{aligned} \pi^{-1}(D) &= \{(x_1, x_2)\} + \check{T} \times \check{T} + \mathrm{Lie}(H) \times \mathrm{Lie}(H) \\ &= \{(x_1, x_2)\} + \check{T}/\check{H} \times \check{T}/\check{H} + \mathrm{Lie}(H) \times \mathrm{Lie}(H). \end{aligned} \tag{10}$$

Suppose that (x_1, x_2) and (y_1, y_2) lie in different components of $\pi^{-1}(D)$. Then $\check{T}/\check{H}$ is nontrivial, and we have

$$(x_1, x_2) = (x_1 + m_1 + h_1, x_2 + m_2 + h_2) \quad \text{and} \quad (y_1, y_2) = (x_1 + m_1' + h_1', x_2 + m_2' + h_2')$$

for some $h_1, h_2, h_1', h_2' \in$Lie(H) and distinct $(m_1, m_2), (m_1', m_2') \in \check{T}/\check{H} \times \check{T}/\check{H}$.

Let $d = d_{\mathfrak{t}\times\mathfrak{t}}$ be the metric induced by the Euclidean inner product on $\mathfrak{t} \times \mathfrak{t}$. We have

$$\begin{aligned} d((x_1, x_2), (y_1, y_2)) &= |(x_1, x_2) - (y_1, y_2)| \\ &= |(m_1 - m_1' + h_1 - h_1', m_2 - m_2' + h_2 - h_2')| \\ &\geqslant |m_1 - m_1', m_2 - m_2'| \end{aligned}$$

where the inequality holds since $\check{T}/\check{H}$ is orthogonal to Lie(H). Since $(m_1, m_2) \neq (m_1', m_2')$, we have

$$d((x_1, x_2), (y_1, y_2)) \geqslant 1.$$

By the first property of an adapted cover, U_{a_1,a_2} is small, which means that V_{x_1,x_2} is contained in the interior of a ball at (x_1, x_2) with radius $\frac{1}{2}$. The same is true for V_{y_1,y_2}. But this means that V_{x_1,x_2} and V_{y_1,y_2} cannot intersect, since $d((x_1, x_2), (y_1, y_2)) \geqslant 1$, so we have a contradiction. Therefore, (x_1, x_2) and (y_1, y_2) must lie in the same connected component of $\pi^{-1}(D)$, and hence the same connected component of $\pi^{-1}(H \times H)$. □

Definition 5.9. Let $\mathcal{U}$ be an open cover of $T \times T$ adapted to $\mathcal{S}$. For each $(t, x) \in \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}$, writing $x = x_1 t_1 + x_2 t_2$, we define an open subset

$$V_{t,x} := \xi_T(\mathcal{X}^+ \times V_{x_1,x_2}) \subset \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}},$$

so that $(t,x) \in V_{t,x}$. Note that $V_{t,x} = V_{t',x'}$ whenever we have $x = x_1t_1 + x_2t_2$ and $x' = x_1t'_1 + x_2t'_2$. The set

$$\{V_{t,x}\}_{(t,x)\in\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}}$$

is an open cover of $\mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}$ (with some redundant elements). The set

$$\{\zeta_T(V_{t,x})\}_{(t,x)\in\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}}$$

is an open cover of E_T.

Definition 5.10. Let $\mathcal{U}$ be an open cover of $T \times T$ adapted to $\mathcal{S}$. Given $a \in E_{T,t}$, define the open subset

$$U_a := \zeta_T(V_{t,x}) \cap E_{T,t}$$

where $x \in \mathfrak{t}_\mathbb{C}$ is any element such that $a = \zeta_{T,t}(x)$, so that $a \in U_a$. The set $\{U_a\}_{a\in E_{T,t}}$ is an open cover of $E_{T,t}$.

Lemma 5.11. *Let $\mathcal{U}$ be an open cover of $T \times T$ which is adapted to $\mathcal{S}$. The open cover of $E_{T,t}$ in Definition 5.10 is adapted to $\mathcal{S}$ in the sense of Definition 3.4.*

Proof. We have

$$\begin{aligned} U_a &:= \zeta_T(V_{t,x}) \cap E_{T,t} \\ &= (\zeta_T \circ \xi_T)(\mathcal{X}^+ \times V_{x_1,x_2}) \cap E_{T,t} \\ &= (\chi_T \circ (\mathrm{id}_{\mathcal{X}^+} \times \pi_{T\times T}))(\mathcal{X}^+ \times V_{x_1,x_2}) \cap E_{T,t} \\ &= \chi_T(\mathcal{X}^+ \times U_{a_1,a_2}) \cap E_{T,t} \\ &= \chi_{T,t}(U_{a_1,a_2}) \end{aligned}$$

where $a_i = \pi_T(x_i)$. Therefore, the open cover $\{U_a\}_{a\in E_{T,t}}$ of $E_{T,t}$ corresponds exactly to the open cover $\mathcal{U}$ of $T \times T$ via the isomorphism

$$\chi_{T,t} : T \times T \cong E_{T,t}$$

of real Lie groups. We have identifications

$$H \times H \cong \mathrm{Hom}(\hat{H}, \mathbb{T})^2 \cong \mathrm{Hom}(\hat{H}, \mathbb{T}^2) \cong \mathrm{Hom}(\hat{H}, E_t) = E_{H,t}$$

and a commutative diagram

$$\begin{array}{ccc} H \times H & \xrightarrow{\cong} & E_{H,t} \\ \big\downarrow & & \big\downarrow \\ T \times T & \xrightarrow{\chi_{T,t}} & E_{T,t}. \end{array} \tag{11}$$

From the diagram, it is clear that the properties of an adapted cover in the sense of Definition 3.4 are equivalent to the properties in Definition 5.3. Since $\mathcal{U}$ is adapted to $\mathcal{S}$ in the sense of Definition 5.3, the result now follows. □

Example 5.12. Let $T = \mathbb{R}/\mathbb{Z}$, and for $\lambda \in \hat{T}$ set X equal to the representation sphere S_λ associated to λ (see Example 1.6). Recall that X has a T-CW complex structure with a 0-cell $T/T \times \mathcal{D}^0_N$ at the north pole, a 0-cell $T/T \times \mathcal{D}^0_S$ at the south pole, and a 1-cell $T \times \mathcal{D}^1$ of free orbits connecting the two poles. Thus,

$$\mathcal{S}(X) = \{T, 1\}.$$

The relations $\leqslant_{\mathcal{S}(X)}$ between the points of $T \times T$ are:

– $(0,0) \leqslant (a_1, a_2)$ for all $(a_1, a_2) \in T \times T$; and
– $(a_1, a_2) \leqslant (b_1, b_2)$ for all $(a_1, a_2), (b_1, b_2) \in T \times T - \{(0,0)\}$.

As in the proof of Lemma 5.4, we can easily construct an open cover of $T \times T$ which is adapted to $\mathcal{S}(X)$. Note that $\mathcal{S}^0 = \mathcal{S}$ in this case. Let $\mathcal{U}$ denote the open cover consisting of open balls U_{a_1,a_2} centered at (a_1, a_2) with radius

$$r_{a_1,a_2} = \begin{cases} \frac{1}{2}\sqrt{a_1^2 + a_2^2} & \text{if } a_1 \neq 0 \text{ or } a_2 \neq 0 \\ \frac{1}{2} & \text{if } (a_1, a_2) = (0,0) \end{cases},$$

where we have identified a_1, a_2 with their unique representatives in $[0,1)$. It is easily verified that $\mathcal{U}$ is a cover adapted to $\mathcal{S}(X)$. Now, for $a = \chi_{T,t}(a_1, a_2)$ and $x = x_1t_1 + x_2t_2$ such that $\zeta_{T,t}(x) = a$, the open set

$$V_{x_1,x_2} \subset \mathbb{R}^2$$

is an open ball of radius $\frac{1}{2}$ if $(x_1, x_2) \in \mathbb{Z}^2$, and an open neighbourhood not intersecting $\mathbb{Z}^2$ otherwise.

6. The Borel-equivariant cohomology of double loop spaces

In this section, we fix a finite T-CW complex X. The double loop space $L^2X = \mathrm{Map}(\mathbb{T}^2, X)$ of X comes equipped with an action of $\mathbb{T}^2 \times T$ via the action of $\mathcal{G}$. We begin with a remark.

Remark 6.1. By Theorem 1.1 in [9], the double loop space $L^2X = \mathrm{Map}(\mathbb{T}^2, X)$ of X is weakly $\mathbb{T}^2 \times T$-homotopy equivalent to a $\mathbb{T}^2 \times T$-CW complex Z.[2] From now on, when we speak of a $\mathbb{T}^2 \times T$-CW structure on L^2X, we mean the replacement complex Z. In this situation, we will abuse notation and write L^2X for Z.

We are going to construct a holomorphic sheaf over $\mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}$ from the $\mathbb{T}^2 \times T$-equivariant cohomology of L^2X. In fact, since we want holomorphic coefficients, we actually consider the holomorphic cohomology rings of all finite subcomplexes of L^2X, and then take the inverse limit over these. The nature of this sheaf is at first obscure, but it will become clearer when, later in the section, we obtain local results which allow us to discard all but a very special class of finite subcomplexes. The following definition is inspired by Definition 3.2 in Kitchloo's paper [8].

Definition 6.2. Define the sheaf of $\mathcal{O}_{\mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}}$-algebras

$$\mathcal{H}^*_{L^2T}(L^2X) := \varprojlim_{Y \subset L^2X} \mathcal{H}^*_{\mathbb{T}^2 \times T}(Y)_{\mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}}$$

where the inverse limit runs over all finite $\mathbb{T}^2 \times T$-CW subcomplexes Y of L^2X.

Remark 6.3. The inverse limit in the category of sheaves is computed in the category of presheaves. Therefore, the value of the inverse limit sheaf $\mathcal{H}^*_{L^2T}(L^2X)$ on an open subset $U \subset \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}$ is

$$\varprojlim_{Y \subset L^2X} H^*_{\mathbb{T}^2 \times T}(Y) \otimes_{H_{\mathbb{T}^2 \times T}} \mathcal{O}_{\mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}}(U).$$

However, it is not true in general that the stalk of the inverse limit is the inverse limit of the stalks.

[2]To apply this theorem to our situation, set $G = \mathbb{T}^2 \times T$. Now let G act on $\mathbb{T}^2$ via the projection to the first factor, and on X via the projection to the second factor.

REMARK 6.4. If $X = \mathrm{pt}$ then $\mathcal{H}^*_{L^2T}(L^2X) = \mathcal{O}_{\mathcal{X}^+\times \mathfrak{t}_\mathbb{C}}$, by construction.

The sheaf $\mathcal{H}^*_{L^2T}(L^2X)$ depends a priori on all finite subcomplexes Y of L^2X, which are of arbitrarily large dimension. Since this is difficult to analyse, it is useful to apply Theorem 2.9 to describe the stalk of $\mathcal{H}^*_{\mathbb{T}^2\times T}(Y)_{(t,x)}$ in terms of the cohomology of the subspace $Y^{t,x} \subset Y$ of loops fixed by a certain subgroup $T(t,x) \subset \mathbb{T}^2 \times T$ (cf. Theorem 3.3 in [8]). Doing this for each Y, we can then describe $\mathcal{H}^*_{L^2T}(L^2X)_{(t,x)}$ as an inverse limit over a class of subspaces of L^2X which is hopefully more tractable than before. It turns out that this is a fruitful line of attack because the subgroup $T(t,x)$ is big enough to fix only a relatively small number of loops in L^2X. Furthermore, we will show that this description holds not only on stalks, but also on the open sets $\{V_{t,x}\}$ defined in the previous section. To achieve this, we need some technical results about the groups $T(t,x)$, the spaces $Y^{t,x}$ and the open sets $V_{t,x}$.

DEFINITION 6.5. Let $(t,x) \in \mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}$. Define $T(t,x)$ to be the intersection

$$T(t,x) = \bigcap_{(t,x)\in \mathrm{Lie}(H)_\mathbb{C}} H$$

of closed subgroups $H \subset \mathbb{T}^2 \times T$. For a $\mathbb{T}^2 \times T$-space Y, denote by $Y^{t,x}$ the subspace of points fixed by $T(t,x)$.

REMARK 6.6. We give an explicit description of $T(t,x)$. Let $(t,x) \in \mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}$ and write $x = x_1t_1 + x_2t_2$. For a closed subgroup $H \subset \mathbb{T}^2 \times T$, the definition of $\mathcal{X}^+$ implies that

$$\begin{aligned}(t,x) = (t_1, t_2, x_1t_1 + x_2t_2) &= (1,0,x_1)t_1 + (0,1,x_2)t_2\\ &\in \mathrm{Lie}(H)\otimes_\mathbb{R}\mathbb{C} = \mathrm{Lie}(H)t_1 + \mathrm{Lie}(H)t_2\end{aligned}$$

if and only if

$$(1,0,x_1),(0,1,x_2) \in \mathrm{Lie}(H).$$

By definition of $T(t,x)$, we therefore have

$$T(t,x) = \langle \pi((r_1,0,x_1r_1)), \pi((0,r_2,x_2r_2))\rangle_{r_1,r_2\in\mathbb{R}}.$$

Since the intersection of $T(t,x)$ with $T \subset \mathbb{T}^2 \times T$ consists exactly of those elements of $T(t,x)$ for which $r_1, r_2 \in \mathbb{Z}$, we have

$$T(t,x) \cap T = \langle \pi(x_1), \pi(x_2)\rangle.$$

REMARK 6.7. Recall the subgroup $T(a) \subset T$ of Definition 3.1. Let $a = \zeta_T(t,x)$ and write $x = x_1t_1 + x_2t_2$. By diagrams (9) and (11), it is clear that $E_{H,t}$ contains a if and only if

$$\pi(x_1), \pi(x_2) \in H.$$

Therefore, $T(a)$ is the closed subgroup

$$\langle \pi(x_1), \pi(x_2)\rangle \subset T,$$

and by Remark 6.6, we have that $T(a) = T \cap T(t,x)$.

LEMMA 6.8. *Let $\zeta_T(t,x) = a$. There is a short exact sequence of compact abelian groups*

$$0 \to T(a) \to T(t,x) \to \mathbb{T}^2 \to 0$$

where $T(t,x) \to \mathbb{T}^2$ is the map induced by the projection $\mathbb{T}^2 \times T \twoheadrightarrow \mathbb{T}^2$.

Proof. The projection of $T(t,x) \subset \mathbb{T}^2 \times T$ onto $\mathbb{T}^2$ is surjective by the description in Remark 6.6, and has kernel $T(t,x) \cap T = T(a)$ by Remark 6.7. □

Notation 6.9. Write

$$p_a : T \twoheadrightarrow T/T(a) \qquad \text{and} \qquad p_{t,x} : \mathbb{T}^2 \times T \twoheadrightarrow (\mathbb{T}^2 \times T)/T(t,x)$$

for the quotient maps, and let

$$\iota_{(0,0)} : T \hookrightarrow \mathbb{T}^2 \times T$$

denote the inclusion of the fiber over $(s_1, s_2) \in \mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$.

Remark 6.10. It follows from Lemma 6.8 that there is a commutative diagram

$$\begin{array}{ccccc} T(a) & \hookrightarrow & T & \xrightarrow{p_a} & T/T(a) \\ \downarrow & & \downarrow \iota_{(0,0)} & & \downarrow \nu \\ T(t,x) & \hookrightarrow & \mathbb{T}^2 \times T & \xrightarrow{p_{t,x}} & (\mathbb{T}^2 \times T)/T(t,x) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{T}^2 & = & \mathbb{T}^2 & \longrightarrow & 0 \end{array} \tag{12}$$

where ν is induced by $\iota_{(0,0)}$.

Lemma 6.11. *The map ν of diagram* (12) *is an isomorphism.*

Proof. By Lemma 6.8, the left hand column is a short exact sequence. It is clear that the middle column is a short exact sequence, as are all rows. It now follows from a standard diagram chase that ν is an isomorphism. □

Remark 6.12. The description of $T(t,x)$ in Remark 6.6 allows us to explicitly describe $L^2X^{t,x}$, using $x = x_1t_1 + x_2t_2$. By this description, a loop $\gamma \in L^2X$ is fixed by $T(t,x)$ if and only if

$$\gamma(s_1, s_2) = \pi((r_1, 0, x_1r_1)) \cdot \gamma(s_1, s_2) = \pi(x_1r_1) \cdot \gamma(s_1 - \pi(r_1), s_2)$$

and

$$\gamma(s_1, s_2) = \pi((0, r_2, x_2r_2)) \cdot \gamma(s_1, s_2) = \pi(x_2r_2) \cdot \gamma(s_1, s_2 - \pi(r_2))$$

for all $s = (s_1, s_2) \in \mathbb{T}^2$ and all $r_1, r_2 \in \mathbb{R}$. By setting $s_1 = s_2 = 0$, one sees that this is true if and only if

$$\gamma(s_1, s_2) = \pi(x_1s_1 + x_2s_2) \cdot \gamma(0, 0),$$

where $\gamma(0,0) \in X^a$ for $a = \zeta_T(t,x)$. Therefore, we have

$$L^2X^{t,x} = \{\gamma \in L^2X \mid \gamma(s_1, s_2) = \pi(x_1s_1 + x_2s_2) \cdot z,\, z \in X^a\}.$$

Although s_1, s_2 are elements of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the loop $\pi(x_1s_1 + x_2s_2) \cdot z$ is well defined since π is a homomorphism and z is fixed by both $\pi(x_1)$ and $\pi(x_2)$. This description should be compared to Corollary 3.4 in [8].

Remark 6.13. The map

$$ev : L^2X \longrightarrow X$$

given by evaluating a loop at $(0,0)$ is evidently T-equivariant and continuous.

Lemma 6.14. *The map ev induces a homeomorphism*

$$ev_{t,x} : L^2X^{t,x} \xrightarrow{\cong} X^a$$

which is natural in X, and equivariant with respect to ν.

Proof. This is evident by Remark 6.12. □

Example 6.15. We calculate $LX^{t,x}$ in the example of the representation sphere $X = S_\lambda$ associated to $\lambda \in \hat{T} = \mathbb{Z}$. For $a \in E_{T,t}$, let a_1, a_2 denote the preimage of a under $\chi_{T,t}$. Then

$$T(a) = \langle a_1, a_2 \rangle \subset T,$$

and we have

$$X^a = \begin{cases} \{N, S\} & \text{if } (a_1, a_2) \neq (0,0) \\ S_\lambda & \text{if } (a_1, a_2) = (0,0) \end{cases}.$$

Let $x \in \zeta_{T,t}^{-1}(a)$ and write $x = x_1 t_1 + x_2 t_2$. We have

$$L^2 X^{t,x} = \begin{cases} \{N, S\} & \text{if } (x_1, x_2) \notin \mathbb{Z}^2 \\ \{\pi(x_1 s_1 + x_2 s_2) \cdot z \,|\, z \in S_\lambda\} & \text{if } (x_1, x_2) \in \mathbb{Z}^2 \end{cases}$$

where N, S denote constant loops. A double loop $\pi(x_1 s_1 + x_2 s_2) \cdot_\lambda z$, for $z \neq N, S$, wraps the first loop around the sphere λx_1 times, and the second loop around λx_2 times, where the loops run parallel to the equator. The direction of the loop corresponds to the sign of the λx_i.

Lemma 6.16. *Let $\mathcal{U}$ be an open cover of $T \times T$ adapted to $\mathcal{S}(X)$. Let $(t,x), (t',y) \in \mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}$, with $a = \zeta_T(t,x)$ and $b = \zeta_T(t',y)$. We have the following.*

(i) If $V_{t,x} \cap V_{t',y} \neq \emptyset$, then either $X^b \subset X^a$ or $X^a \subset X^b$.

(ii) If $V_{t,x} \cap V_{t',y} \neq \emptyset$ and $X^b \subset X^a$, then $L^2 X^{t',y} \subset L^2 X^{t,x}$.

Proof. Write $x = x_1 t_1 + x_2 t_2$, $y = y_1 t'_1 + y_2 t'_2$ and $a_i = \pi(x_i)$, $b_i = \pi(y_i)$. Since $V_{t,x} \cap V_{t',y} \neq \emptyset$, we have

$$V_{x_1,x_2} \cap V_{y_1,y_2} \neq \emptyset$$

by definition. This implies that

$$U_{a_1,a_2} \cap U_{b_1,b_2} \neq \emptyset,$$

so by the second property of an adapted cover, either $(a_1, a_2) \leqslant (b_1, b_2)$ or $(b_1, b_2) \leqslant (a_1, a_2)$. This implies that either $X^b \subset X^a$ or $X^a \subset X^b$, which yields the first part of the result.

For the second part, assume that $X^b \subset X^a$, and let $\gamma \in L^2 X^{t',y}$. Let $z = \gamma(0,0)$. By the description in Remark 6.12, we have

$$\gamma(s_1, s_2) = \pi(y_1 s_1 + y_2 s_2) \cdot z.$$

Let $H \subset T$ be the isotropy group of $z \in X^b \subset X^a$, so that $H \in \mathcal{S}(X)$ and $a_i, b_i \in H$ for $i = 1, 2$. The condition

$$V_{x_1,x_2} \cap V_{y_1,y_2} \neq \emptyset$$

implies by Lemma 5.8 that (x_1, x_2) and (y_1, y_2) lie in the same component of $\pi^{-1}(H \times H)$, since $\mathcal{U}$ is adapted to $\mathcal{S}(X)$. Therefore, $(x_1 - y_1, x_2 - y_2)$ lies in the identity component of $\pi^{-1}(H \times H)$, which is equal to $\mathrm{Lie}(H) \times \mathrm{Lie}(H)$. This implies that z is fixed by $\pi((x_1 - y_1) r_1)$ and $\pi((x_2 - y_2) r_2)$ for all $r_1, r_2 \in \mathbb{R}$.

We can now write

$$\begin{aligned} \gamma(s_1, s_2) &= \pi(y_1 s_1 + y_2 s_2) \cdot z \\ &= \pi(y_1 s_1 + y_2 s_2) \cdot (\pi((x_1 - y_1) s_1 + (x_2 - y_2) s_2) \cdot z) \\ &= \pi(x_1 s_1 + x_2 s_2) \cdot z, \end{aligned}$$

which is a loop in $L^2 X^{t,x}$ since $z \in X^a$. This yields the second part of the result. □

Lemma 6.17. *Let $\mathcal{U}$ be an open cover adapted to $\mathcal{S}(X)$. If $(t', y) \in V_{t,x}$, then $L^2X^{t',y} \subset L^2X^{t,x}$.*

Proof. Write $a = \zeta_T(t, x)$, $b = \zeta_T(t', y)$, $x = x_1t_1 + x_2t_2$, $y = y_1t_1' + y_2t_2'$, $b_i = \pi(y_i)$, and $a_i = \pi(x_i)$. Since $(t', y) \in V_{t,x}$, we have

$$(y_1, y_2) \in V_{x_1,x_2} \subset \mathfrak{t} \times \mathfrak{t}$$

and so

$$(b_1, b_2) \in U_{a_1,a_2} \subset T \times T.$$

Therefore $(a_1, a_2) \leqslant (b_1, b_2)$ by Lemma 5.6, from which it follows that $X^b \subset X^a$, since $\mathcal{U}$ is adapted to $\mathcal{S}(X)$. Lemma 6.16 yields the result. □

Example 6.18. We examine the inclusions of Lemmas 6.16 and 6.17 in the example of the representation sphere $X = S_\lambda$. Let $(t, x), (t', y) \in \mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}$, write $x = x_1t_1 + x_2t_2$, $y = y_1t_1' + y_2t_2'$, $a = \zeta_T(t, x)$ and $b = \zeta_T(t', y)$. If $V_{t,x} \cap V_{t,y} = \emptyset$, then by Definition 5.9 we have

$$V_{x_1,x_2} \cap V_{y_1,y_2} \neq \emptyset.$$

Therefore, by Example 5.12, at least one of (x_1, x_2) and (y_1, y_2) lies outside the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. By Example 6.15, this means that either $L^2X^{t',y} = X^b = \{N, S\}$ or $L^2X^{t,x} = X^a = \{N, S\}$, and we clearly have either $X^b \subset X^a$ or $X^a \subset X^b$. If we assume that $X^b \subset X^a$, then since at least one of the spaces is equal to $\{N, S\}$, we must have $X^b = \{N, S\}$. Therefore, $L^2X^{t',y} \subset L^2X^{t,x}$. Note that if we had the additional hypothesis that $(t', y) \in V_{t,x}$, then this would imply that $(y_1, y_2) \in V_{x_1,x_2}$, and by the description in Example 5.12 we would have $(y_1, y_2) \notin \mathbb{Z}^2$, which means that $X^b \subset X^a$. So, with this hypothesis, no assumption would be necessary.

Proposition 6.19. *Let $Y \subset L^2X$ be a finite $\mathbb{T}^2 \times T$-CW subcomplex. The inclusion $Y^{t,x} \subset Y$ induces an isomorphism of $\mathcal{O}_{V_{t,x}}$-algebras*

$$\mathcal{H}^*_{\mathbb{T}^2\times T}(Y)_{V_{t,x}} \cong \mathcal{H}^*_{\mathbb{T}^2\times T}(Y^{t,x})_{V_{t,x}}.$$

Proof. Let $(t', y) \in V_{t,x}$. Then $L^2X^{t',y} \subset L^2X^{t,x}$ by Lemma 6.17, which implies that $Y^{t',y} \subset Y^{t,x}$. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^*_{\mathbb{T}^2\times T}(Y)_{V_{t,x}} & \longrightarrow & \mathcal{H}^*_{\mathbb{T}^2\times T}(Y^{t,x})_{V_{t,x}} \\ & \searrow \quad \swarrow & \\ & \mathcal{H}^*_{\mathbb{T}^2\times T}(Y^{t',y})_{V_{t,x}}, & \end{array} \tag{13}$$

which is induced by the evident inclusions. Taking stalks at (t', y), Theorem 2.9 implies that the two diagonal maps are isomorphisms, and so the horizontal map is also an isomorphism. The isomorphism of the proposition follows. □

Remark 6.20. The subspace $L^2X^{t,x} \subset L^2X$ is a $\mathbb{T}^2 \times T$-equivariant CW subcomplex, consisting of those equivariant cells in L^2X whose isotropy group contains $T(t, x)$. In fact, it follows easily from Lemma 6.14 that $L^2X^{t,x}$ is a finite $\mathbb{T}^2 \times T$-CW complex, since X is finite.

Corollary 6.21. *The inclusion $L^2X^{t,x} \subset L^2X$ induces an isomorphism of $\mathcal{O}_{V_{t,x}}$-algebras*

$$\mathcal{H}^*_{L^2T}(L^2X)_{V_{t,x}} \cong \mathcal{H}^*_{\mathbb{T}^2\times T}(L^2X^{t,x})_{V_{t,x}},$$

natural in X. In particular, we have an isomorphism of stalks

$$\mathcal{H}^*_{L^2T}(L^2X)_{(t,x)} \cong \mathcal{H}^*_{\mathbb{T}^2\times T}(L^2X^{t,x})_{(t,x)}.$$

Proof. It follows from Definition 6.2 and Proposition 6.19 that

$$\begin{aligned}\mathcal{H}^*_{L^2T}(L^2X)_{V_{t,x}} &= \varprojlim_{Y\subset L^2X} \mathcal{H}^*_{\mathbb{T}^2\times T}(Y)_{V_{t,x}} \\ &\cong \varprojlim_{Y\subset L^2X} \mathcal{H}^*_{\mathbb{T}^2\times T}(Y^{t,x})_{V_{t,x}} \\ &= \mathcal{H}^*_{\mathbb{T}^2\times T}(L^2X^{t,x})_{V_{t,x}}.\end{aligned}$$

The final equality holds by definition of the inverse limit, since each $Y^{t,x}$ is contained in the finite $\mathbb{T}^2 \times T$-CW subcomplex $L^2X^{t,x} \subset L^2X$. One can easily show naturality with respect to a T-equivariant map $f : X \to Y$ by refining the cover $\mathcal{U}$ so that it is adapted to $\mathcal{S}(f)$, and by using the functoriality of the loop space functor and of Borel-equivariant ordinary cohomology. The second statement follows immediately by definition of the stalk. □

Remark 6.22. It follows from Corollary 6.21 that $\mathcal{H}^*_{L^2T}(L^2X)$ is a coherent sheaf, since $L^2X^{t,x}$ is a finite $\mathbb{T}^2 \times T$-CW complex.

7. The construction of the equivariant sheaf $\mathcal{E}^*_T(X)$ over E_T

In this section, we begin with a result which says that $\mathcal{H}_{L^2T}(L^2X)$ depends only on loops which are contained in a subspace of the form $L^2X^{t,x}$, even if we are computing global sections. This is an important feature of our construction, as it allows us to do without a $\mathbb{T}^2 \times T$-CW complex structure on L^2X, and also makes computations much more tractable. We then recall the $\mathcal{G}$-action on L^2X and explain how this induces an action of the Weyl group $\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2$ on $\mathcal{H}_{L^2T}(L^2X)$, which also carries a natural $\mathbb{C}^\times$-action. Finally, we obtain the $\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z})$-equivariant sheaf $\mathcal{E}^*_T(X)$ over E_T as the $\check{T}^2$-invariants of the pushforward of $\mathcal{H}_{L^2T}(L^2X)$ along ζ_T.

Definition 7.1. Let X be a finite T-CW complex, and let $\mathcal{D}^2(X)$ denote the set of finite $\mathbb{T}^2 \times T$-CW subcomplexes of L^2X generated by

$$\{L^2X^{t,x}\}_{(t,x)\in\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}}$$

under finite unions and intersections. The set $\mathcal{D}^2(X)$ is partially ordered by inclusion.

Theorem 7.2. *Let X be a finite T-CW complex. There is an isomorphism of $\mathcal{O}_{\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}}$-algebras*

$$\mathcal{H}^*_{L^2T}(L^2X) \cong \varprojlim_{Y\in\mathcal{D}^2(X)} \mathcal{H}^*_{\mathbb{T}^2\times T}(Y)_{\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}}$$

natural in X.

Proof. Consider the union

$$S := \bigcup_{(t,x)\in\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}} L^2X^{t,x}.$$

Notice that S is a $\mathbb{T}^2 \times T$-CW subcomplex of L^2X. For each finite $\mathbb{T}^2 \times T$-CW subcomplex $Y \subset L^2X$, we have an inclusion $Y \cap S \hookrightarrow Y$. The induced map of $\mathcal{O}_{\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}}$-algebras

$$\varprojlim_{Y\subset L^2X} \mathcal{H}^*_{\mathbb{T}^2\times T}(Y)_{\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}} \to \varprojlim_{Y\subset L^2X} \mathcal{H}^*_{\mathbb{T}^2\times T}(Y\cap S)_{\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}} \tag{14}$$

is natural in X, by the functoriality of Borel-equivariant ordinary cohomology. Let (t,x) be an arbitrary point. By Corollary 6.21, the map (14) induces an isomorphism of stalks at (t,x) because S contains $L^2X^{t,x}$. Therefore, the map (14) is an isomorphism of sheaves.

It is clear that the set $\{S \cap Y \mid Y \subset L^2X \text{ finite}\}$ is equal to the set of all finite equivariant subcomplexes of S. Therefore, the target of map (14) is equal to the inverse limit

$$\varprojlim_{Y\subset S} \mathcal{H}^*_{\mathbb{T}^2\times T}(Y)_{\mathcal{X}^+\times \mathfrak{t}_{\mathbb{C}}} \tag{15}$$

over all finite equivariant subcomplexes $Y \subset S$. Any such Y, since it is finite, is contained in the union of finitely many spaces of the form $L^2X^{t,x}$, which is also a finite $\mathbb{T}^2 \times T$-CW complex. Therefore, by definition of the inverse limit, (15) is equal to

$$\varprojlim_{Y\in\mathcal{D}^2(X)} \mathcal{H}^*_{\mathbb{T}^2\times T}(Y)_{\mathcal{X}^+\times \mathfrak{t}_{\mathbb{C}}},$$

which completes the proof. □

Theorem 7.3. *Let X be a finite T-CW complex. The sheaf $\mathcal{H}^*_{L^2T}(L^2X)$ is a $\mathbb{C}^\times \times (\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2)$-equivariant sheaf of $\mathcal{O}_{\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}}$-algebras.*

Proof. First, we describe the $\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2$-action on $\mathcal{H}_{L^2T}(L^2X)$. Recall that $\mathcal{G}$ acts on L^2X via

$$(A, t, \gamma(s)) \cdot \gamma'(s) = \gamma(A^{-1}s - At) \cdot \gamma'(A^{-1}s - At).$$

Define a left action of $\mathcal{G}$ on $E\mathcal{G} \times L^2X$ by

$$g \cdot (e, \gamma) = (e \cdot g^{-1}, g \cdot \gamma)$$

for all $g \in \mathcal{G}$ and $(e, \gamma) \in E\mathcal{G} \times L^2X$. Note that the maximal torus $\mathbb{T}^2 \times T$ of $\mathcal{G}$ acts freely on $E\mathcal{G}$ via the action of $\mathcal{G}$. Therefore, the quotient of $E\mathcal{G} \times L^2X$ by the $\mathbb{T}^2 \times T$-action is a model for the $\mathbb{T}^2 \times T$-equivariant Borel construction

$$\mathbb{T}^2 \times T\backslash E\mathcal{G} \times L^2X \cong E(\mathbb{T}^2 \times T) \times_{\mathbb{T}^2\times T} L^2X.$$

Recall that $\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2$ is the Weyl group associated to $\mathbb{T}^2 \times T \subset \mathcal{G}$. The action of $\mathcal{G}$ on $E\mathcal{G} \times L^2X$ therefore induces an action

$$\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2 \quad \circlearrowright \quad E(\mathbb{T}^2 \times T) \times_{\mathbb{T}^2\times T} L^2X.$$

Passing to the subspace of fixed loops, the action of $(A, m) \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2$ induces a homeomorphism

$$E(\mathbb{T}^2 \times T) \times_{\mathbb{T}^2\times T} L^2X^{t,x} \longrightarrow E(\mathbb{T}^2 \times T) \times_{\mathbb{T}^2\times T} L^2X^{At,x+mt}$$

for each $(t, x) \in \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}$. Writing $x = x_1t_1 + x_2t_2$, the homeomorphism sends

$$(e, \pi(x_1s_1 + x_2s_2) \cdot z) \mapsto (e \cdot (A, m)^{-1}, \pi((x_1 + m_1, x_2 + m_2)A^{-1}(s_1, s_2)) \cdot z). \tag{16}$$

Note that

$$\pi((x_1 + m_1, x_2 + m_2)A^{-1}(s_1, s_2)) \cdot z$$

does in fact lie in $L^2X^{At,x+mt}$, since

$$x + mt = ((x_1 + m_1)t_1, (x_2 + m_2)t_2) = (x_1 + m_1, x_2 + m_2)A^{-1}A(t_1, t_2).$$

There is an induced isomorphism on cohomology rings

$$(A, m)^* : H^*_{\mathbb{T}^2\times T}(L^2X^{At,x+mt}) \longrightarrow H^*_{\mathbb{T}^2\times T}(L^2X^{t,x}).$$

Let $\omega_{A,m} : \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}} \to \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}$ be the action map of (A, m). Then $(A, m)^*$ induces an isomorphism of sheaves

$$\omega^*_{A,m}(\mathcal{H}^*_{\mathbb{T}^2\times T}(L^2X^{At,x+mt})_{\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}}) \longrightarrow \mathcal{H}^*_{\mathbb{T}^2\times T}(L^2X^{t,x})_{\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}}.$$

In fact, we can get such a map for each $Y \in \mathcal{D}^2(X)$ by replacing $L^2X^{t,x}$ with Y and $L^2X^{At,x+mt}$ with $(A,m)\cdot Y$. It is easily verified that $(A,m)\cdot Y \in \mathcal{D}^2(X)$ whenever $Y \in \mathcal{D}^2(X)$. Now, the family of isomorphisms

$$\{\omega^*_{A,m}(\mathcal{H}^*_{\mathbb{T}^2\times T}((A,m)\cdot Y)_{\mathcal{X}^+\times \mathfrak{t}_{\mathbb{C}}}) \longrightarrow \mathcal{H}^*_{\mathbb{T}^2\times T}(Y)_{\mathcal{X}^+\times \mathfrak{t}_{\mathbb{C}}}\}_{Y\in\mathcal{D}^2(X)}$$

is compatible with inclusions in $\mathcal{D}^2(X)$, since they are induced by the action of $\mathcal{G}$ on L^2X. We therefore have an induced isomorphism

$$\omega^*_{A,m}\mathcal{H}^*_{L^2T}(L^2X) \longrightarrow \mathcal{H}^*_{L^2T}(L^2X)$$

of inverse limit sheaves, which induces an isomorphism

$$(A,m)^* : \mathcal{H}^*_{\mathbb{T}^2\times T}(L^2X^{At,x+mt})_{(At,x+mt)} \longrightarrow \mathcal{H}^*_{\mathbb{T}^2\times T}(L^2X^{t,x})_{(t,x)} \tag{17}$$

of stalks. It is straightforward to verify that the collection of isomorphisms

$$\{\omega^*_{A,m}\mathcal{H}^*_{L^2T}(L^2X) \longrightarrow \mathcal{H}^*_{L^2T}(L^2X)\}_{(A,m)\in \mathrm{SL}_2(\mathbb{Z})\ltimes \check{T}^2}$$

defines an action of $\mathrm{SL}_2(\mathbb{Z})\ltimes \check{T}^2$ on $\mathcal{H}^*_{L^2T}(L^2X)$.

We now describe the $\mathbb{C}^\times$-action on $\mathcal{H}^*_{L^2T}(L^2X)$. Let $\mathbb{C}^\times$ act on $\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}$ by

$$\lambda\cdot(t,x) = (\lambda^2 t, \lambda^2 x).$$

This is to be compared with the scalar action on $\mathcal{X}^+$ in Remark 1.3. Here, we introduce the square because the holomorphic coordinate functions have degree 2 when regarded as classes in equivariant cohomology. The $\mathbb{C}^\times$-action on the sheaf for each $\lambda\in\mathbb{C}^\times$ is induced by an isomorphism

$$\begin{array}{rcl} \omega^*_\lambda(\mathcal{H}^*_{\mathbb{T}^2\times T}(L^2X^{\lambda^2 t,\lambda^2 x})_{\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}}) & \longrightarrow & \mathcal{H}^*_{\mathbb{T}^2\times T}(L^2X^{t,x})_{\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}} \\ \alpha\otimes f & \longmapsto & \lambda^n\alpha\otimes f \end{array}$$

for each $(t,x)\in\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}$. Here, α is a degree n cohomology class and ω_λ denotes the action of λ on $\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}$. On an open subset $U\subset\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}$, the isomorphism is

$$\begin{array}{rcl} H^*_{\mathbb{T}^2\times T}(L^2X^{\lambda^2 t,\lambda^2 x})_{\lambda^2 U} & \longrightarrow & H^*_{\mathbb{T}^2\times T}(L^2X^{t,x})_U \\ \alpha\otimes f & \longmapsto & \lambda\cdot\alpha\otimes f\circ\omega_\lambda. \end{array}$$

Thus, if α is a class of degree i and f satisfies $f(\lambda^2 t,\lambda^2 x) = \lambda^j f(t,x)$, then $\alpha\otimes f$ is sent to $\lambda^{i+j}(\alpha\otimes f)$. It is straightforward to verify that this is an action on $\mathcal{H}^*_{L^2T}(L^2X)$ and that it commutes with the action of $\mathrm{SL}_2(\mathbb{Z})\ltimes\check{T}^2$. □

REMARK 7.4. We denote by ι_t the inclusion of the fiber $\mathfrak{t}_{\mathbb{C}}\hookrightarrow\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}$ over $t\in\mathcal{X}^+$. We have a commutative diagram of complex manifolds

$$\begin{array}{ccc} \mathfrak{t}_{\mathbb{C}} & \xhookrightarrow{\iota_t} & \mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}} \\ \Big\downarrow{\scriptstyle \zeta_{T,t}} & & \Big\downarrow{\scriptstyle \zeta_T} \\ E_{T,t} & \hookrightarrow & E_T. \end{array}$$

The $\mathbb{C}^\times\times\mathrm{SL}_2(\mathbb{Z})$-action on E_T does not preserve the fiber over t.

The following definition is analogous to Definition 4.1 in Kitchloo's paper [8].

DEFINITION 7.5. Let X be a finite T-CW complex. Define the $\mathbb{C}^\times\times\mathrm{SL}_2(\mathbb{Z})$-equivariant, coherent sheaf of $\mathcal{O}_{E_T}$-algebras

$$\mathcal{E}^*_T(X) := ((\zeta_T)_*\,\mathcal{H}^*_{L^2T}(L^2X))^{\check{T}^2},$$

which has a $\mathbb{Z}$-grading corresponding to the action of $\mathbb{C}^\times$. Define the coherent sheaf of $\mathcal{O}_{E_{T,t}}$-algebras

$$\mathcal{E}^*_{T,t}(X) := ((\zeta_{T,t})_* \, \iota_t^* \, \mathcal{H}^*_{L^2T}(L^2X))^{\check{T}^2},$$

which has a $\mathbb{Z}/2\mathbb{Z}$-grading given by decomposing into even and odd cohomology degrees.

REMARK 7.6. The $\mathbb{C}^\times$-action on $\mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}$ by $\lambda \cdot (t,x) = \lambda^2(t,x)$ induces a grading on the structure sheaf $\mathcal{O}^*_{\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}}$ whereby $f : U \to \mathbb{C}$ lies in $\mathcal{O}^j_{\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}}(U)$ if and only if $f(\lambda^2 t, \lambda^2 x) = \lambda^j f(t,x)$ for all $\lambda \in \mathbb{C}^\times, t \in U$. This grading is clearly compatible with the $\mathrm{SL}_2(\mathbb{Z}) \ltimes \check{T}^2$-action. By construction, we have that $\mathcal{E}^*_T(\mathrm{pt})$ is equal to the $\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z})$-equivariant structure sheaf

$$\mathcal{O}^*_{E_T} = ((\zeta_T)_* \mathcal{O}^*_{\mathcal{X}^+\times\mathfrak{t}_\mathbb{C}})^{\check{T}^2}$$

of E_T. Similarly, $\mathcal{E}^*_{T,t}(\mathrm{pt})$ is equal to the structure sheaf

$$\mathcal{O}_{E_{T,t}} = ((\zeta_{T,t})_* \mathcal{O}_{\mathfrak{t}_\mathbb{C}})^{\check{T}^2}$$

of the fiber of E_T over $t \in \mathcal{X}^+$. Note that $\mathcal{O}_{E_{T,t}}$ has trivial grading since the scalar action does not preserve $\{t\} \times \mathfrak{t}_\mathbb{C} \subset \mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}$.

EXAMPLE 7.7. It is straightforward to compute $\mathcal{E}^*_T(X)$ when $T = 1$, for this implies that $\mathcal{D}^2(X) = \{X\}$. We obtain the $\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z})$-equivariant sheaf whose value on an open subset $U \subset \mathcal{X}^+$ is

$$\mathcal{E}^*_1(X)(U) = H^*(X) \otimes_\mathbb{C} \mathcal{O}^*_{\mathcal{X}^+}(U).$$

The value of $\mathcal{E}^n_1(X)$ on U is the vector space generated by elements of the form $\alpha \otimes f$, where $\alpha \in H^i(X)$, $f \in \mathcal{O}^j(U)$ and $i + j = n$.

EXAMPLE 7.8. In the case that $T = 1$ and $X = \mathrm{pt}$, we obtain the $\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{Z})$-equivariant structure sheaf $\mathcal{O}^*_{\mathcal{X}^+}$. The $\mathrm{SL}_2(\mathbb{Z})$-invariant global sections of $\mathcal{E}^n_1(\mathrm{pt}) = \mathcal{O}^n_{\mathcal{X}^+}$ are exactly the weak modular forms of weight $-n/2$. In particular, the $\mathrm{SL}_2(\mathbb{Z})$-invariant global sections of $\mathcal{E}^*_1(\mathrm{pt})$ are concentrated in negative even degrees.

8. The suspension isomorphism

In this section, we use the description of $\mathcal{H}^*_{L^2T}(L^2X)$ given in Theorem 7.2 to show that the reduced theory $\widetilde{\mathcal{E}}^*_T(X)$ inherits a suspension isomorphism from ordinary cohomology.

REMARK 8.1. Let X be a finite T-CW complex. We regard the loop space $L^2(S^1 \wedge X_+)$ as a pointed $\mathbb{T}^2 \times T$-CW complex with basepoint given by the loop $\gamma_* : \mathbb{T}^2 \to \mathrm{pt} \hookrightarrow S^1 \wedge X_+$. Since the basepoint of $S^1 \wedge X_+$ is fixed by T, the loop γ_* is fixed by $\mathbb{T}^2 \times T$, and so γ_* is contained in $L^2(S^1 \wedge X_+)^{t,x}$ for all (t,x). Therefore, each $Y \in \mathcal{D}^2(S^1 \wedge X_+)$ is a based subcomplex of $L^2(S^1 \wedge X_+)$.

LEMMA 8.2. *For each $(t,x) \in \mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}$, we have an equality*

$$L^2(S^1 \wedge X_+)^{t,x} = S^1 \wedge L^2 X_+^{t,x}$$

as subsets of $L^2(S^1 \wedge X_+)$.

Proof. Suppose that γ is a loop in $L^2(S^1 \times X_+)^{t,x}$ sending $s \mapsto (\gamma_1(s), \gamma_2(s))$. Write $\gamma(0) = (z_1, z_2)$ and $x = x_1 t_1 + x_2 t_2$. The loop γ is fixed by $T(t,x)$ if and only if

$$(\gamma_1(s), \gamma_2(s)) = \pi(x_1 r_1 + x_2 r_2) \cdot (\gamma_1(s-r), \gamma_2(s-r)) = (\gamma_1(s-r), \pi(x_1 r_1 + x_2 r_2) \cdot \gamma_2(s-r))$$

for all $r, s \in \mathbb{T}^2$, where S^1 is fixed by T. Setting $r = s$, one sees that this is true if and only if

$$(\gamma_1(r), \gamma_2(r)) = (\gamma_1(0), \pi(x_1 r_1 + x_2 r_2) \cdot \gamma_2(0)),$$

which holds if and only γ_1 is constant in S^1 and γ_2 is in $L^2 X_+^{t,x}$. Therefore, we have an equality

$$L^2(S^1 \times X_+)^{t,x} = S^1 \times L^2 X_+^{t,x}.$$

To prove the equality of the lemma, consider that the image of γ is contained in $0 \times X_+ \cup S^1 \times \mathrm{pt}$ if and only if we have either $\gamma_1(s) = 0$ or $\gamma_2(s) = \mathrm{pt}$ for each s. But this means that either $\gamma_1(s) = 0$ for all s, or $\gamma_2(s) = \mathrm{pt}$ for all s, which means that $\gamma \in 0 \times L^2 X^{t,x} \cup S^1 \times \mathrm{pt}$. The equality of the lemma follows. □

DEFINITION 8.3. For a pointed finite T-CW complex X, define the reduced theory

$$\widetilde{\mathcal{E}}_T^*(X) := \ker(\mathcal{E}_T^*(X) \to \mathcal{E}_T^*(\mathrm{pt})).$$

PROPOSITION 8.4. *Let X be a finite T-CW complex. There is an $\mathrm{SL}_2(\mathbb{Z})$-equivariant isomorphism of sheaves of $\mathcal{O}_{E_T}$-modules*

$$\mathcal{E}_T^{*-1}(X) \cong \widetilde{\mathcal{E}}_T^*(S^1 \wedge X_+)$$

natural in X.

Proof. We have

$$\begin{aligned}
\mathcal{E}_T^{*-1}(X) &:= ((\zeta_T)_* \varprojlim_{Y \in \mathcal{D}^2(X)} \mathcal{H}^{*-1}_{\mathbb{T}^2 \times T}(Y)_{\mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}})^{\check{T}^2} \\
&\cong ((\zeta_T)_* \varprojlim_{Y \in \mathcal{D}^2(X)} \widetilde{\mathcal{H}}^{*}_{\mathbb{T}^2 \times T}(S^1 \wedge Y_+)_{\mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}})^{\check{T}^2} \\
&= ((\zeta_T)_* \varprojlim_{Y \in \mathcal{D}^2(S^1 \wedge X_+)} \widetilde{\mathcal{H}}^{*}_{\mathbb{T}^2 \times T}(Y)_{\mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}})^{\check{T}^2} \\
&=: ((\zeta_T)_* \varprojlim_{Y \in \mathcal{D}^2(S^1 \wedge X_+)} \ker(\mathcal{H}^{*}_{\mathbb{T}^2 \times T}(Y) \to \mathcal{H}^{*}_{\mathbb{T}^2 \times T}(\mathrm{pt}))_{\mathcal{X}^+ \times \mathfrak{t}_\mathbb{C}})^{\check{T}^2} \\
&= \ker(\mathcal{E}_T^*(S^1 \wedge X_+) \to \mathcal{E}_T^*(\mathrm{pt})) \\
&=: \widetilde{\mathcal{E}}_T^*(S^1 \wedge X_+).
\end{aligned}$$

Indeed, the second line is induced by the natural suspension isomorphism of ordinary cohomology. The third holds because the equality $L^2(S^1 \wedge X_+)^{t,x} = S^1 \wedge L^2 X_+^{t,x}$ of Lemma 8.2 is an equality of subsets of $L^2(S^1 \wedge X_+)$ for each (t, x) which are compatible with inclusion, so that they extend to an equality of partially ordered sets

$$\mathcal{D}^2(S^1 \wedge X_+) = \{S^1 \wedge Y_+\}_{Y \in \mathcal{D}^2(X)}.$$

The fifth holds since the inverse limit is a right adjoint functor, and therefore respects all limits, including kernels. The remaining equalities hold by definition. The isomorphism of the second line is $\mathrm{SL}_2(Z)$-equivariant by the functoriality of Borel-equivariant cohomology, and each equality is evidently $\mathrm{SL}_2(\mathbb{Z})$-equivariant. Therefore, the composite is $\mathrm{SL}_2(\mathbb{Z})$-equivariant. □

COROLLARY 8.5. *The association $X \mapsto \mathcal{E}_T^*(X)$ induces a functor from the category of finite T-CW complexes into the category of $\mathbb{Z}$-graded coherent $\mathcal{O}_{\mathcal{M}_T}$-algebras, which is a reduced cohomology theory when it is equipped with the suspension isomorphisms of Proposition 8.4.*

Proof. The association of $\mathcal{E}_T^*(X)$ to X is clearly functorial and homotopy invariant, since both the double loop space functor and Borel-equivariant ordinary cohomology are functorial and homotopy invariant. It is also exact and additive, since these properties may be checked on stalks, and Corollary 6.21 implies that these properties are inherited from Borel-equivariant ordinary cohomology (since $L^2 X^{t,x} \cong X^a$ by Lemma 6.14; compare the proof of Proposition 3.16). □

9. A calculation of $\mathcal{E}_T^*(T/H)$

In this section, we again use Theorem 7.2 to compute $\mathcal{E}_T^*(X)$ for an orbit $X = T/H$, where the T-action is induced by group multiplication. Namely, we will show that

$$\mathcal{E}_T^*(T/H) = \mathcal{O}_{E_H},$$

where E_H is the image of $\mathcal{X}^+ \times H \times H$ under the isomorphism $\chi_T : \mathcal{X}^+ \times T \times T \longrightarrow E_T$. One may then compute $\mathcal{E}_T^*(X)$ for any other finite T-CW complex X by using the Mayer-Vietoris sequence.

We begin by calculating $\mathcal{D}^2(T/H)$. Let $a \in E_{T,t}$ and write (a_1, a_2) for the preimage of a under $\chi_{T,t}$. We have

$$(T/H)^a = (T/H)^{\langle a_1, a_2 \rangle} = \begin{cases} T/H & \text{if } (a_1, a_2) \in H \times H \\ \emptyset & \text{otherwise.} \end{cases}.$$

Let $x \in \zeta_{T,t}^{-1}(a)$ and write $x = x_1 t_1 + x_2 t_2$. We have that

$$L^2 X^{t,x} = \{\gamma(s_1, s_2) = \pi(x_1 s_1 + x_2 s_2) \cdot z \,|\, z \in T/H\} \tag{18}$$

if $(x_1, x_2) \in \pi^{-1}(H \times H)$, and $L^2 X^{t,x}$ is empty otherwise. To calculate $\mathcal{D}^2(T/H)$, we need to calculate the intersections of these subspaces. Suppose

$$\gamma(s_1, s_2) \in L^2 X^{t,x} \cap L^2 X^{t',x'}$$

with $x = x_1 t_1 + x_2 t_2$ and $x' = x_1' t_1' + x_2' t_2'$. Then

$$\gamma(s_1, s_2) = \pi(x_1 s_1 + x_2 s_2) \cdot z = \pi(x_1' s_1 + x_2' s_2) \cdot z'$$

for $z, z' \in T/H$. A straightforward calculation shows that this holds if and only if

$$z - z' = \pi((x_1 - x_1')s_1 + (x_2 - x_2')s_2) \bmod H$$

for all s_1, s_2, which holds if and only if

$$z = z' \quad \text{and} \quad (x_1 - x_1', x_2 - x_2') \in \mathrm{Lie}(H) \times \mathrm{Lie}(H).$$

Thus, for all $(t, x), (t', x') \in \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}$, the intersection

$$L^2 X^{t,x} \cap L^2 X^{t',x'} = L^2 X^{t,x} = L^2 X^{t',x'},$$

if and only if $(x_1, x_2), (x_1', x_2')$ lie in the same component of $\pi^{-1}(H \times H)$, and is empty otherwise. In particular, the spaces $L^2 X^{t,x}$ are indexed by the components of $\pi^{-1}(H \times H)$, where the component containing (x_1, x_2) has the form

$$\{(x_1, x_2)\} + \mathrm{Lie}(H) \times \mathrm{Lie}(H) \subset \mathfrak{t} \times \mathfrak{t}.$$

We will call (x_1, x_2) a *representative* of this component. Moreover, we have seen that there are no nonempty intersections between two spaces of the form $L^2 X^{t,x}$, unless they are equal. Therefore $\mathcal{D}^2(X)$ is the set of finite disjoint unions of such spaces.

Before moving on to the next step, which is to calculate the cohomology ring $H_{\mathbb{T}^2 \times T}(L^2 X^{t,x})$, we need to know more about the space $L^2 X^{t,x}$. Since the $\mathbb{T}^2 \times T$-action on $L^2 X^{t,x}$ is clearly transitive, we can apply the change of groups property of Proposition 2.1 if we know the $\mathbb{T}^2 \times T$-isotropy. An element $(r, u) \in \mathbb{T}^2 \times T$ fixes a nonempty subset $L^2 X^{t,x}$ if and only if

$$(r, u) \cdot (\pi(x_1 s_1 + x_2 s_2) \cdot z) = (\pi(x_1(s_1 - r_1) + x_2(s_2 - r_2)) + u) \cdot z = \pi(x_1 s_1 + x_2 s_2) \cdot z,$$

which holds if and only if $\pi(-x_1r_1 - x_2r_2) + u$ fixes z. Therefore, we must have

$$u - \pi(x_1r_1 + x_2r_2) \in H,$$

which means that the isotropy group of the $\mathbb{T}^2 \times T$-orbit $L^2X^{t,x}$ is equal to

$$\{(r, u) \in \mathbb{T}^2 \times T \,|\, u - \pi(x_1r_1 + x_2r_2) \in H\} = \langle T(t, x), H\rangle. \tag{19}$$

Furthermore, since two spaces of the form $L^2X^{t,x}$ are equal if they correspond to the same component of $\pi^{-1}(H \times H)$, we must have that

$$\langle T(t, x), H\rangle = \langle T(t', x'), H\rangle$$

whenever $(x_1, x_2), (x'_1, x'_2)$ lie in the same component $\pi^{-1}(H \times H)$. Using Proposition 2.1, we calculate

$$H_{\mathbb{T}^2\times T}(L^2X^{t,x}) \cong H_{\mathbb{T}^2\times T}((\mathbb{T}^2 \times T)/\langle T(t, x), H\rangle) \cong H_{\langle T(t,x),H\rangle}.$$

The value of $\mathcal{H}^*_{L^2T}(L^2X)$ on an open subset $U \subset \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}}$ is therefore

$$\begin{aligned}&\varprojlim_{Y\in\mathcal{D}^2(X)} H^*_{\mathbb{T}^2\times T}(Y) \otimes_{H_{\mathbb{T}^2\times T}} \mathcal{O}_{\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}}(U)\\ &= \prod_{(x_1,x_2)\in J(H)} H_{\langle T(x_1,x_2),H\rangle} \otimes_{H_{\mathbb{T}^2\times T}} \mathcal{O}_{\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}}(U)\end{aligned}$$

where the product is indexed over a set $J(H) = \{(x_1, x_2)\}$ of representatives of the components of $\pi^{-1}(H \times H)$, and $T(x_1, x_2) := T(t, x)$ for any (t, x) such that $x = x_1t_1 + x_2t_2$.

From the description in (19), we see that

$$\mathrm{Lie}(\langle T(x_1, x_2), H\rangle)_{\mathbb{C}} = \{(t, y) \in \mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}} \,|\, y \in (x_1 + \mathrm{Lie}(H))t_1 + (x_2 + \mathrm{Lie}(H))t_2\}. \tag{20}$$

Let $I(x_1, x_2, H) \subset \mathbb{C}[t_1, t_2, y]$ be the ideal associated to $\mathrm{Lie}(\langle T(x_1, x_2), H\rangle)_{\mathbb{C}} \subset \mathbb{C}^2 \times \mathfrak{t}_{\mathbb{C}}$, so that

$$H_{\langle T(x_1,x_2),H\rangle} = \mathbb{C}[t, y]/I(x_1, x_2, H).$$

By Proposition 2.8 in [12], tensoring over $H_{\mathbb{T}^2\times T}$ with the ring of holomorphic functions is an exact functor. Therefore, if $\mathcal{I}(x_1, x_2, H) \subset \mathcal{O}_{\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}}$ denotes the analytic ideal associated to $I(x_1, x_2, H)$, we can write

$$\mathcal{H}^*_{L^2T}(L^2(T/H)) = \prod_{(x_1,x_2)\in J(H)} \mathcal{O}_{\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}}/\mathcal{I}(x_1, x_2, H). \tag{21}$$

Example 9.1. If T has rank one and $H = \mathbb{Z}/n\mathbb{Z}$, then $\mathrm{Lie}(H)$ is trivial and we have

$$\mathcal{H}^*_{L^2T}(L^2(T/H))(U) = \prod_{(x_1,x_2)} \mathbb{C}[t_1, t_2, y]/(y - x_1t_1 - x_2t_2) \otimes_{\mathbb{C}[t,y]} \mathcal{O}_{\mathcal{X}^+\times\mathfrak{t}_{\mathbb{C}}}(U)$$

where (x_1, x_2) ranges over $J(\mathbb{Z}/n\mathbb{Z}) = \{(a_1/n, a_2/n) \,|\, a_1, a_2 \in \mathbb{Z}\}$. This is a holomorphic version of the calculation made by Rezk in Example 5.2 of [10].

Now, the $\check{T}^2$-action induces isomorphisms

$$\mathcal{O}(\mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}})/\mathcal{I}(x_1 + m_1, x_2 + m_2, H) \longrightarrow \mathcal{O}(\mathcal{X}^+ \times \mathfrak{t}_{\mathbb{C}})/\mathcal{I}(x_1, x_2, H),$$

for each $m \in \check{T}^2$ and each $(x_1, x_2) \in J(H)$, given by pullback along $(t, y) \mapsto (t, y + mt)$. We calculate the invariants by first noting that

$$\pi^{-1}(H \times H) = \coprod_{(x_1,x_2)\in J(H)} \{(x_1, x_2)\} + \mathrm{Lie}(H) \times \mathrm{Lie}(H).$$

Thus, the diagram

$$\begin{array}{ccc}
\mathcal{X}^+ \times \pi^{-1}(H\times H) & \xrightarrow{\cong} & \coprod_{(x_1,x_2)\in J(H)} \mathrm{Lie}(\langle T(x_1,x_2), H\rangle)_{\mathbb{C}} \\
\downarrow & & \downarrow \\
\mathcal{X}^+ \times H \times H & \xrightarrow{\cong} & E_H
\end{array}$$

commutes, where the arrows are the evident restrictions of the maps appearing in the diagram (9), and $E_H \subset E_T$ is defined to be the image of $\mathcal{X}^+ \times H \times H$ under χ_T. Since the right vertical map is exactly the preimage under ζ_T of E_H, we see that E_H is the complex analytic quotient of the free $\check{T}^2$ action on the complex vector space (20), which has holomorphic structure sheaf (21). Therefore, the $\check{T}^2$-invariants of (21) is equal to the sheaf of holomorphic functions on E_H, and so we have that

$$\mathcal{E}^*_T(T/H) = \mathcal{O}_{E_H}.$$

10. A local description

In this section, we give a local description of $\mathcal{E}_{T,t}(X)$ over any cover of $E_{T,t}$ which is adapted to X. This local description turns out to be identical to Grojnowski's construction of the equivariant elliptic cohomology of X corresponding to the elliptic curve E_t, as outlined in Section 3.

Notation 10.1. Let $K(t,x) := (\mathbb{T}^2 \times T)/T(t,x)$. Let $t_{t,x}$ denote translation by (t,x) in $\mathbb{C}^2 \times \mathfrak{t}_{\mathbb{C}}$, and t_x translation by x in $\mathfrak{t}_{\mathbb{C}}$. If f is a map of compact Lie groups, we abuse notation and also write f for the induced map of complex Lie algebras.

Remark 10.2. It will be essential to the proof of the following Lemma to show that the diagram

$$\begin{array}{ccc}
\mathfrak{t}_{\mathbb{C}} & \xrightarrow{p_a \circ t_{-x}} & \mathrm{Lie}(T/T(a))_{\mathbb{C}} \\
\downarrow \iota_t & & \cong \downarrow \nu \\
\mathbb{C}^2 \times \mathfrak{t}_{\mathbb{C}} & \xrightarrow{p_{t,x}} & \mathrm{Lie}(K(t,x))_{\mathbb{C}}
\end{array} \tag{22}$$

commutes.

Lemma 10.3. *There is an isomorphism of sheaves of $\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}$-algebras*

$$t^*_{-x}\, p^*_a\, \mathcal{H}^*_{T/T(a)}(X^a) \cong \iota^*_t\, p^*_{t,x}\, \mathcal{H}^*_{K(t,x)}(L^2X^{t,x})$$

natural in X.

Proof. Recall the isomorphism $\nu : K(t,x) \cong T/T(a)$ of diagram (12). By Lemma 6.14, the evaluation map

$$ev_{t,x} : L^2X^{t,x} \cong X^a$$

is natural in X and equivariant with respect to ν. The induced homeomorphism

$$E(K(t,x)) \times_{K(t,x)} L^2X^{t,x} \cong E(T/T(a)) \times_{T/T(a)} X^a$$

induces, in turn, an isomorphism of $H_{T/T(a)}$-algebras

$$H^*_{T/T(a)}(X^a) \cong H^*_{K(t,x)}(L^2X^{t,x}),$$

natural in X. Here the ring $H_{T/T(a)}$ acts on the target via the isomorphism

$$H_{K(t,x)} \cong H_{T/T(a)}$$

induced by ν. Thus, we have an isomorphism of $\mathcal{O}_{\mathrm{Lie}(T/T(a))_{\mathbb{C}}}$-algebras

$$\mathcal{H}^*_{T/T(a)}(X^a) \cong \nu^* \mathcal{H}^*_{K(t,x)}(L^2 X^{t,x}),$$

and hence an isomorphism of $\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}$-algebras,

$$t^*_{-x}\, p^*_a\, \mathcal{H}^*_{T/T(a)}(X^a) \cong t^*_{-x}\, p^*_a\, \nu^*\, \mathcal{H}^*_{K(t,x)}(L^2 X^{t,x})$$

natural in X.

We now show that diagram (22) commutes, from which the result follows immediately. The commutative diagram (12) induces a commutative diagram of complex Lie algebras, so that

$$\nu \circ p_a = p_{t,x} \circ \iota_{(0,0)}$$

on complex Lie algebras. Consider the commutative diagram

$$\begin{array}{ccc} T(t,x) & \longrightarrow & \mathbb{T}^2 \times T \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & (\mathbb{T}^2 \times T)/T(t,x) \end{array}$$

of compact abelian groups. By applying the Lie algebra functor and then tensoring with $\mathbb{C}$, we see that (t,x) lies in the kernel of

$$p_{t,x} : \mathbb{C}^2 \times \mathfrak{t}_{\mathbb{C}} \twoheadrightarrow \mathrm{Lie}((\mathbb{T}^2 \times T)/T(t,x))_{\mathbb{C}}.$$

Therefore,

$$p_{t,x} = p_{t,x} \circ t_{t,x}$$

and so

$$\begin{aligned} \nu \circ p_a &= p_{t,x} \circ t_{t,x} \circ \iota_{(0,0)} \\ &= p_{t,x} \circ \iota_t \circ t_x. \end{aligned}$$

Composing on the right by t_{-x} now yields the commutativity of diagram (22). □

Theorem 10.4. *Let X be a finite T-CW complex and let $\mathcal{U}$ be a cover adapted to $\mathcal{S}(X)$. Let $a \in E_{T,t}$, and let U_a be the corresponding open neighbourhood of a in $E_{T,t}$. There is an isomorphism of sheaves of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{O}_{U_a}$-algebras*

$$\mathcal{E}^*_{T,t}(X)_{U_a} \cong \mathcal{G}^*_{T,t}(X)_{U_a},$$

natural in X.

Proof. For $x \in \zeta^{-1}_{T,t}(a)$, write V_x for the component of $\zeta^{-1}_{T,t}(U_a)$ containing x. We have a sequence of isomorphisms

$$\begin{aligned} ((\zeta_{T,t})_* \iota^*_t \mathcal{H}^*_{L^2 T}(L^2 X))_{U_a} &\cong \prod_{x \in \zeta^{-1}_{T,t}(a)} (\iota^*_t \mathcal{H}^*_{\mathbb{T}^2 \times T}(L^2 X^{t,x}))_{V_x} \\ &\cong \prod_{x \in \zeta^{-1}_{T,t}(a)} (\iota^*_t\, p^*_{t,x} \mathcal{H}^*_{K(t,x)}(L^2 X^{t,x}))_{V_x} \\ &\cong \prod_{x \in \zeta^{-1}_{T,t}(a)} (t^*_{-x}\, p^*_a\, \mathcal{H}^*_{T/T(a)}(X^a))_{V_x} \\ &\cong \prod_{x \in \zeta^{-1}_{T,t}(a)} (t^*_{-x}\, \mathcal{H}^*_T(X^a))_{V_x} \end{aligned} \tag{23}$$

The first map is the isomorphism of Corollary 6.19. The second map and fourth maps are the isomorphism of Proposition 2.2. The third map is the isomorphism of Lemma 10.3. The first to the fourth maps, in each factor, have been shown to preserve the $\mathcal{O}_{V_x}$-algebra structure. The entire composite is therefore an isomorphism of $\mathcal{O}_{U_a}$-algebras, where $f \in \mathcal{O}_{U_a}(U)$ acts on the right hand side via multiplication by $\zeta^*_{T,t} f$. Each map has been shown to be natural in X, replacing, if necessary, the covering $\mathcal{U}$ with a refinement $\mathcal{U}(f)$ adapted to a T-equivariant map $f : X \to Y$. All maps preserve the $\mathbb{Z}$-grading on the cohomology, except the map of Proposition 2.2. The latter does, however, preserve the $\mathbb{Z}/2\mathbb{Z}$-grading by odd and even elements, since it is defined by taking the cup product with elements of H_T, which have even degree.

It remains to find the image of the $\check{T}^2$-invariants under the composite map above. Let $U \subset U_a$ be an open subset and let $V \subset V_x$ be an open set such that $V \cong U$ via $\zeta_{T,t}$. Over U, the composite is equal to

$$\begin{aligned} \prod H_{L^2T}(L^2X)_{\{t\}\times V} &\cong \prod H_{\mathbb{T}^2\times T}(L^2X^{t,x})_{\{t\}\times V} \\ &\cong \prod H_{K(t,x)}(L^2X^{t,x})_{\{t\}\times V} \\ &\cong \prod H_{T/T(a)}(X^a)_{V-x} \\ &\cong \prod H_T(X^a)_{V-x} \end{aligned}$$

where each product runs over all $x \in \zeta_{T,t}^{-1}(a)$. Note that

$$V_x - x = V_{x+mt} - x - mt$$

for all m. We will show

$$\left\{ \prod_{x\,\in\,\zeta_{T,t}^{-1}(a)} H_T(X^a)_{V-x} \right\}^{\check{T}^2} = H_T(X^a)_{V-x}$$

by showing that $\check{T}^2$ merely permutes the indexing set of the product. This yields our result via the isomorphism

$$\mathcal{G}^*_{T,t}(X)_{U_a}(U) = H^*_T(X^a) \otimes_{H_T} \mathcal{O}_{E_{T,t}}(U-a) \cong H^*_T(X^a) \otimes_{H_T} \mathcal{O}_{\mathfrak{t}_\mathbb{C}}(V-x) =: H_T(X^a)_{V-x}$$

induced by the canonical isomorphism $V - x \cong U - a$, as in Remark 3.7.

In other words, we must show that the action of m induces the identity map

$$H_{T/T(a)}(X^a)_{V_{x+mt}-x-mt} = H_{T/T(a)}(X^a)_{V_x-x}.$$

To do this, it suffices to check the commutativity of two diagrams. The first diagram is

$$\begin{array}{ccc} H_{\mathbb{T}^2\times T}(L^2X^{t,x+mt})_{\{t\}\times V_{x+mt}} & \longrightarrow & H_{K(t,x+mt)}(L^2X^{t,x+mt})_{\{t\}\times V_{x+mt}} \\ \downarrow m^* & & \downarrow m^* \\ H_{\mathbb{T}^2\times T}(L^2X^{t,x})_{\{t\}\times V_x} & \longrightarrow & H_{K(t,x)}(L^2X^{t,x})_{\{t\}\times V_x} \end{array} \tag{24}$$

where the vertical arrows are induced by the action of m on the spaces

$$E(\mathbb{T}^2\times T)\times_{\mathbb{T}^2\times T} L^2X^{t,x} \longrightarrow E(\mathbb{T}^2\times T)\times_{\mathbb{T}^2\times T} L^2X^{t,x+mt}$$

and

$$E(K(t,x))\times_{K(t,x)} L^2X^{t,x} \longrightarrow E(K(t,x+mt))\times_{K(t,x+mt)} L^2X^{t,x+mt}.$$

The horizontal maps are defined, as in Proposition 2.2, using the Eilenberg-Moore spectral sequences associated to the pullback diagrams

$$\begin{array}{ccc} E(\mathbb{T}^2 \times T) \times_{\mathbb{T}^2\times T} L^2X^{t,x} & \longrightarrow & B(\mathbb{T}^2 \times T) \\ \downarrow & & \downarrow \\ E(K(t,x)) \times_{K(t,x)} L^2X^{t,x} & \longrightarrow & B(K(t,x)) \end{array} \tag{25}$$

and

$$\begin{array}{ccc} E(\mathbb{T}^2 \times T) \times_{\mathbb{T}^2\times T} L^2X^{t,x+mt} & \longrightarrow & B(\mathbb{T}^2 \times T) \\ \downarrow & & \downarrow \\ E(K(t,x+mt)) \times_{K(t,x+mt)} L^2X^{t,x+mt} & \longrightarrow & B(K(t,x+mt)). \end{array} \tag{26}$$

It is easily verified that the action of m induces an isomorphism from diagram (25) to diagram (26), from which it follows that (24) commutes.

The second diagram to check is

$$\begin{array}{ccc} H_{K(t,x+mt)}(L^2X^{t,x+mt})_{\{t\}\times V_{x+mt}} & \longrightarrow & H_{T/T(a)}(X^a)_{V_{x+mt-x-mt}} \\ \Big\downarrow{\scriptstyle m^*} & & \Big\| \\ H_{K(t,x)}(L^2X^{t,x})_{\{t\}\times V_x} & \longrightarrow & H_{T/T(a)}(X^a)_{V_x-x} \end{array} \tag{27}$$

with vertical maps induced by the action of m, ane horizontal maps as in the proof of Lemma 10.3. By the proof of Lemma 10.3, diagram (27) commutes if

$$\begin{array}{ccc} E(K(t,x)) \times_{K(t,x)} L^2X^{t,x} & \xrightarrow{E\nu^{-1}\times ev_{t,x}} & E(T/T(a)) \times_{T/T(a)} X^a \\ \Big\downarrow{\scriptstyle m} & & \Big\| \\ E(K(t,x+mt)) \times_{K(t,x+mt)} L^2X^{t,x+mt} & \xrightarrow{E\nu^{-1}\times ev_{t,x+mt}} & E(T/T(a)) \times_{T/T(a)} X^a \end{array} \tag{28}$$

commutes, where $ev_{t,x} : L^2X^{t,x} \cong X^a$ is equivariant with respect to $\nu^{-1} : K(t,x) \cong T/T(a)$. To see that this commutes, note firstly that ν is induced by the inclusion $T \hookrightarrow \mathbb{T}^2 \times T$ of the fixed points of the Weyl action $(r,t) \mapsto (r, t+mr)$, which implies that $\nu^{-1} \circ m = \nu^{-1}$. Secondly, note that the action of m on a loop γ fixes $\gamma(0,0)$, which implies that $ev_{t,x+mt} \circ m = ev_{t,x}$. These two observations imply the commutativity of diagram (28), which completes the proof. □

Remark 10.5. Our aim is now to determine the gluing maps associated to the local description in Theorem 10.4. Let X be a finite T-CW complex, let $\mathcal{U}$ be a cover adapted to $\mathcal{S}(X)$, and let $a, b \in E_{T,t}$ be such that $U_a \cap U_b \neq \emptyset$. Choose $x \in \zeta_{T,t}^{-1}(a)$ and $y \in \zeta_{T,t}^{-1}(b)$ such that $V_x \cap V_y \neq \emptyset$, which implies that $V_{t,x} \cap V_{t,y} \neq \emptyset$. It follows from Lemma 6.16 that either $X^b \subset X^a$ or $X^a \subset X^b$. We may assume that $X^b \subset X^a$. Let U be an open subset in $U_a \cap U_b$ and let $V \subset V_x \cap V_y$ be such that $V \cong U$ via $\zeta_{T,t}$. Let H be the subgroup of T generated by $T(a)$ and $T(b)$. Consider the composite of isomorphisms

$$\begin{aligned} H_T(X^a)\otimes_{H_T}\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V-x) &\cong H_T(X^b)\otimes_{H_T}\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V-x) \\ &\cong H_{T/H}(X^b)\otimes_{H_T}\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V-x) \\ &\cong H_{T/H}(X^b)\otimes_{H_T}\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V-y) \\ &\cong H_T(X^b)\otimes_{H_T}\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V-y) \end{aligned}$$

The first map is induced by the inclusion $X^b \subset X^a$, the second map and fourth maps are induced by the change of groups map of Proposition 2.2, and the third map is

$$\mathrm{id}\otimes t^*_{y-x} : H_{T/H}(X^b)\otimes_{H_{T/H}}\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V-x) \longrightarrow H_{T/H}(X^b)\otimes_{H_{T/H}}\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V-y),$$

which is a well defined map of $\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}$-algebras, by Lemma 10.6. All maps thus preserve the $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}$-algebra structure. We have a commutative diagram

$$\begin{array}{ccc} V-x & \xrightarrow{t_{y-x}} & V-y \\ \downarrow{\scriptstyle \zeta_{T,t}} & & \downarrow{\scriptstyle \zeta_{T,t}} \\ U-a & \xrightarrow{t_{b-a}} & U-b \end{array}$$

of complex analytic isomorphisms. Via this diagram, the composite above is canonically identified with

$$\begin{aligned} H_T(X^a)\otimes_{H_T}\mathcal{O}_{E_{T,t}}(V-x) &\cong H_T(X^b)\otimes_{H_T}\mathcal{O}_{E_{T,t}}(U-a) \\ &\cong H_{T/H}(X^b)\otimes_{H_T}\mathcal{O}_{E_{T,t}}(U-a) \\ &\cong H_{T/H}(X^b)\otimes_{H_T}\mathcal{O}_{E_{T,t}}(U-b) \\ &\cong H_T(X^b)\otimes_{H_T}\mathcal{O}_{E_{T,t}}(U-b), \end{aligned}$$

where the fourth map is $\mathrm{id}\otimes t^*_{b-a}$. This is the gluing map $\phi_{b,a}$ of Grojnowski's construction (see Remark 4), and we will show in Theorem 10.7 that this is the gluing map associated to Theorem 10.4.

Lemma 10.6. *With the hypotheses of Remark 10.5, the translation map*

$$t^*_{y-x} : \mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V-x) \longrightarrow \mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V-y)$$

is $H_{T/H}$-linear.

Proof. Let $x = x_1t_1 + x_2t_2$ and $y = y_1t_1 + y_2t_2$. By the same argument as in the proof of Lemma 6.16, we have that

$$(y_1 - x_1, y_2 - x_2) \in \mathrm{Lie}(H)\times \mathrm{Lie}(H),$$

since $T(a), T(b) \subset H$. Therefore

$$y - x = (y_1 - x_1)t_1 + (y_2 - x_2)t_2 \in \mathrm{Lie}(H)\otimes_{\mathbb{R}}(\mathbb{R}t_1 + \mathbb{R}t_2) = \mathrm{Lie}(H)\otimes_{\mathbb{R}}\mathbb{C}.$$

This implies the result. □

Theorem 10.7. *With the hypotheses of Remark 10.5, the gluing map associated to the local description in Theorem 10.4 on an open subset $U \subset U_a \cap U_b$ is equal to the composite map in Remark* (10.5).

Proof. We must first make a few observations before we can make sense of the diagram which we will use to prove the theorem. Let K be the subgroup of $\mathbb{T}^2\times T$ generated by $T(t,x)$ and $T(t,y)$. We have $H \subset T\cap K$, since

$$\begin{aligned} H = \langle T(a), T(b)\rangle &= \langle T\cap T(t,x), T\cap T(t,y)\rangle \\ &\subset T\cap\langle T(t,x), T(t,y)\rangle = T\cap K. \end{aligned}$$

Furthermore, the inclusion $T \subset \mathbb{T}^2 \times T$ induces an isomorphism $T/(T \cap K) \cong (\mathbb{T}^2 \times T)/K$, as may be verified by chasing a diagram analogous to (12). We therefore have identifications

$$X^{T \cap K} = \mathrm{Map}_T(T/(T \cap K), X) \cong \mathrm{Map}_T((\mathbb{T}^2 \times T)/K, X) = (L^2 X)^K = L^2 X^{t,y} \cong X^b,$$

where the mapping spaces $\mathrm{Map}_T(-,-)$ of T-equivariant maps are identified with fixed-point spaces by evaluating at $0 \in T$. The composite is T-equivariant if we let T act on mapping spaces via the target space. Thus, X^b is fixed by $T \cap K$, and the homeomorphism $X^b \cong LX^{t,y}$ is equivariant with respect to $T/(T \cap K) \cong (\mathbb{T}^2 \times T)/K$. Finally, note that Lemma 6.16 implies that

$$\begin{CD} L^2 X^{t,y} @>{i_{y,x}}>> L^2 X^{t,x} \\ @V{\cong}V{ev_{t,y}}V @V{\cong}V{ev_{t,x}}V \\ X^b @>{i_{b,a}}>> X^a \end{CD} \tag{29}$$

commutes. Now, the diagram is as follows.

$$\begin{array}{ccccccc}
H_T(X^a)_{V-x} & \longrightarrow & H_{T/T(a)}(X^a)_{V-x} & \longrightarrow & H_{K(t,x)}(L^2 X^{t,x})_{\{t\} \times V} & \longrightarrow & H_{\mathbb{T}^2 \times T}(L^2 X^{t,x})_{\{t\} \times V} \\
\downarrow & (1) & \downarrow & (2) & \downarrow & (1) & \downarrow \\
H_T(X^b)_{V-x} & \longrightarrow & H_{T/T(a)}(X^b)_{V-x} & \longrightarrow & H_{K(t,x)}(L^2 X^{t,y})_{\{t\} \times V} & \longrightarrow & H_{\mathbb{T}^2 \times T}(L^2 X^{t,y})_{\{t\} \times V} \\
\downarrow & (3) & \downarrow & (4) & \downarrow & (3) & \downarrow \\
H_{T/H}(X^b)_{V-x} & \longrightarrow & H_{T/(T \cap K)}(X^b)_{V-x} & \longrightarrow & H_{(\mathbb{T}^2 \times T)/K}(L^2 X^{t,y})_{\{t\} \times V} & \longrightarrow & H_{K(t,y)}(L^2 X^{t,y})_{\{t\} \times V} \\
\downarrow & (5) & \downarrow & (6)\ \swarrow & (4) & \swarrow & \\
H_{T/H}(X^b)_{V-y} & \longrightarrow & H_{T/(T \cap K)}(X^b)_{V-y} & \longleftarrow & H_{T/T(b)}(X^b)_{V-y} & & \\
\downarrow & (3) & \swarrow & & & & \\
H_T(X^b)_{V-y} & & & & & &
\end{array} \tag{30}$$

Each map is an isomorphism of $\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V)$-algebras, and is exactly one of the following four types:

- the change of groups map of Proposition 2.2 (if the target and source only differ by equivariance group);
- the map induced by an inclusion of spaces (if the target and source only differ by the topological space);
- the translation map $\mathrm{id} \otimes t^*_{y-x}$ of Remark 10.5 (these are the vertical maps of region (5) - note that translation by $y - x$ is $H_{T/(T \cap K)}$-linear, since $H \subset T \cap K$); or
- the map of Lemma 10.3, or a map obtained in a way exactly analogous to the proof of Lemma 10.3 (see below).

If diagram (30) commutes, then the two outermost paths from $H_T(X^a)_{V-x}$ to $H_T(X^b)_{V-y}$

are equal, which yields the result. Indeed, this is true, because each of the numbered regions in diagram (30) commutes for the respective reason stated below.

(1) By naturality of the isomorphism of Proposition 2.2.
(2) By naturality of the isomorphism of Lemma 10.3, along with diagram (29).
(3) By Lemma 2.4.
(4) This holds essentially because the homeomorphism $X^b \cong LX^{t,y}$ is equivariant with respect to the inclusion $T \subset \mathbb{T}^2 \times T$. (See below for a more detailed proof.)
(5) This is obvious.
(6) Note that both (t, x) and (t, y) are contained in the complexified Lie algebra of K, and are therefore also in the kernel of
$$\mathbb{C}^2 \times \mathfrak{t}_{\mathbb{C}} \twoheadrightarrow \mathrm{Lie}((\mathbb{T}^2 \times T)/K)_{\mathbb{C}}.$$
Thus, in the proof of Lemma 10.3, we may translate by either (t, x) or (t, y), leading to the horizontal and diagonal maps, respectively. These two maps evidently commute with the vertical map, which is $\mathrm{id} \otimes t^*_{y-x}$.

In more detail, the claim of item 4 holds by a proof similar to that of Lemma 10.3. Indeed, the identification $X^b \cong LX^{t,y}$ is equivariant with respect to the isomorphisms $T/(T \cap K) \cong (\mathbb{T}^2 \times T)/K$, $T/T(a) \cong K(t, x)$ and $T/T(b) \cong K(t, y)$, which are all induced by the inclusion $T \subset \mathbb{T}^2 \times T$. This implies that, in the case of the middle square, we have an isomorphism of the diagram

$$\begin{array}{ccc} E(T/T(a)) \times_{T/T(a)} X^b & \longrightarrow & B(T/T(a)) \\ \downarrow & & \downarrow \\ E(T/(T \cap K)) \times_{T/(T\cap K)} X^b & \longrightarrow & B(T/(T \cap K)), \end{array} \tag{31}$$

which induces the left vertical map, and the diagram

$$\begin{array}{ccc} E(K(t, x)) \times_{K(t,x)} L^2X^{t,y} & \longrightarrow & B(K(t, x)) \\ \downarrow & & \downarrow \\ E((\mathbb{T}^2 \times T)/K) \times_{(\mathbb{T}^2\times T)/K} L^2X^{t,y} & \longrightarrow & B((\mathbb{T}^2 \times T)/K), \end{array} \tag{32}$$

which induces the right vertical map. We therefore have a commutative square

$$\begin{array}{ccc} H_{T/T(a)}(X^b) & \longrightarrow & H_{K(t,x)}(L^2X^{t,y}) \\ \downarrow & & \downarrow \\ H_{T/(T\cap K)}(X^b) \otimes_{H_{T/(T\cap K)}} H_{T/T(a)} & \longrightarrow & H_{(\mathbb{T}^2\times T)/K}(L^2X^{t,y}) \otimes_{H_{(\mathbb{T}^2\times T)/K}} H_{K(t,x)} \end{array}$$

of isomorphisms of $H_{T/T(a)} \cong H_{K(t,x)}$-algebras. Then, to get isomorphisms of $\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}(V)$-algebras, we tensor the left hand side over $p_a \circ t_{-x}$ and the right hand side over $p_{t,x} \circ \iota_t$, as in diagram (22). One shows the commutativity of the other square labelled (4) in exactly the same way, replacing $T(a)$ with $T(b)$ and $K(t, x)$ with $K(t, y)$. □

Corollary 10.8. *There is an isomorphism of cohomology theories*

$$\mathcal{E}^*_{T,t} \cong \mathcal{G}^*_{T,t},$$

defined on the category of finite T-CW complexes and taking values in the category of sheaves of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{O}_{E_{T,t}}$-algebras.

Proof. The local description of $\mathcal{E}_{T,t}(X)$ given in Theorems 10.4 and 10.7 amounts to the natural isomorphism that we require. It is clear that this is compatible with suspension isomorphisms since, in each theory, these are induced by the suspension isomorphism in ordinary cohomology, by the proofs of Propositions 3.16 and 8.4. □

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Matthew Spong matt.spong@gmail.com
School of Mathematics and Statistics, The University of Melbourne, Parkville VIC 3010, Australia